CHAPTER 5

SINGLE AND TWO-PHASE FLOW MODELLING I

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ABSTRACT

The derivation of the cross section averaged form of the mass, momentum, and energy equations in terms of time-averaged variables is presented. The analysis is based on the fact that the variables may fluctuate as turbulent quantities, and it is shown that the techniques of turbulent flows can be applied to two-phase flows. The one-dimensional single phase turbulent flow conservation equations, and the homogeneous two-phase flow equations can then be obtained by simplification. The homogeneous equilibrium flow model is developed with particular consideration given to the momentum equation and pressure drop.

5.1. INTRODUCTION

The derivation of the conservation equations for one-dimensional two-phase flows is given in detail in the appendix. The following is a brief development of these equations with the emphasis given to the concepts of the control volume (integral) formulation, time averaging of fluctuating (turbulent) variables, and cross-section-area averaging.

5.2. THE GENERAL CONSERVATION EQUATION FOR A CONTROL VOLUME

$$\frac{dX}{dt} = \frac{\partial}{\partial t}\iiint x\,\rho\, dV + \oiint x(\rho \mathbb{V}\cdot d\mathbb{A}) \qquad (1)$$

Rate of Generation = Rate of Increase in Control Volume + Net Efflux Integrated over Control Surface

5.2.1 Conservation of Mass (Continuity Equation)

X ≡ total mass M of system

$$\Gamma_{1m} \equiv \frac{dM}{dt} \equiv \text{the volumetric generation of component 1}$$

From (1)

$$\oiint \rho_1 \alpha \mathbb{V}_1 \cdot d\mathbb{A} + \frac{\partial}{\partial t}\iiint \rho_1 \alpha\, dV = \iiint \tilde{\Gamma}_{1m}\, dV \qquad (2)$$

For turbulent flows, the fluctuating quantities are:

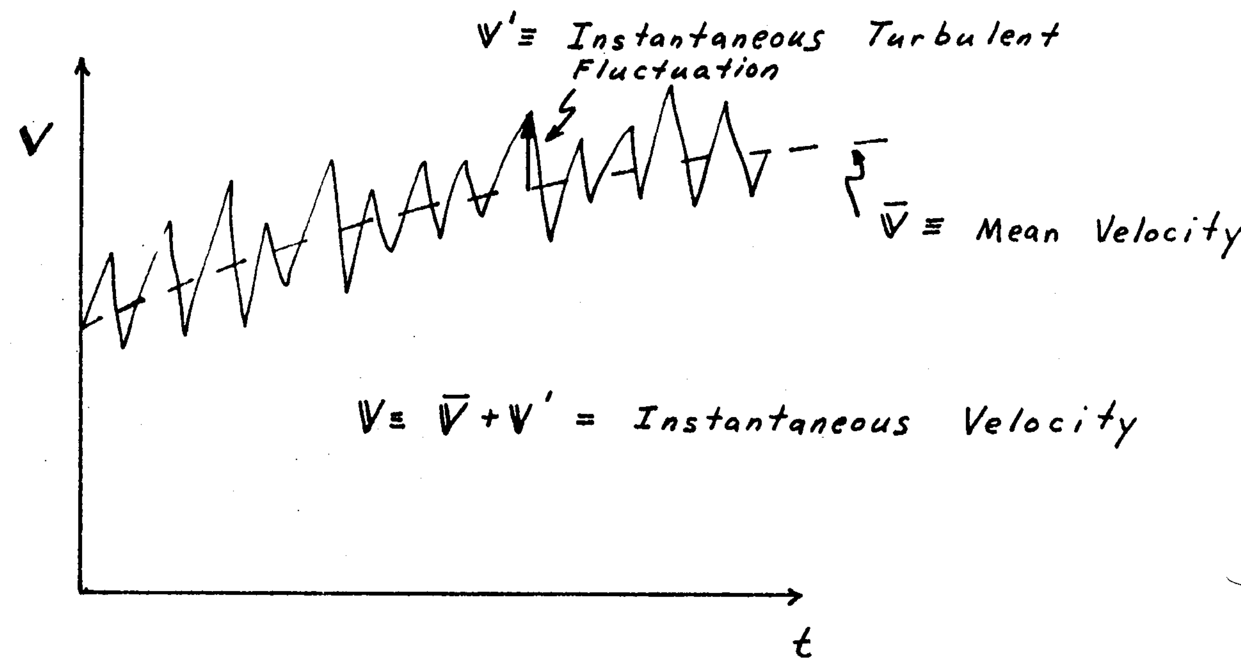
$$\begin{aligned} \alpha &= \bar{\alpha} + \alpha' \\ \rho_1 &= \bar{\rho} + \rho' \\ \mathbb{V}_1 &= \bar{\mathbb{V}} + \mathbb{V}' \\ \tilde{\Gamma}_{1m} &= \bar{\tilde{\Gamma}}_{1m} + \tilde{\Gamma}_{1m}' \end{aligned} \qquad (3)$$

Figure 1 is a sketch showing turbulent fluctuations in velocity $\mathbb{V}$ and void fraction α.

The time-averaging operator is defined as

$$(\overline{}) \equiv \frac{1}{\Delta t} \int_{\Delta t} (\quad) \, dt \tag{4}$$

Hence by definition $\overline{\mathbb{V}'} = 0$, $\overline{\mathbb{V}'^2} \neq 0$



Examples: Hot Wire Anemometer (single phase)
Pitot Tube (Single Phase, Two-Phase)

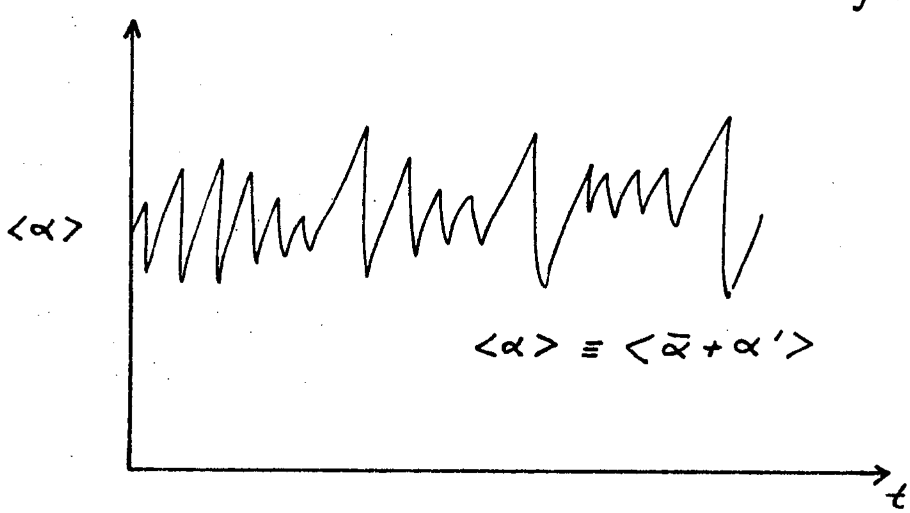


Fig. 1: Examples: Gamma Densitometer
Neutron Absorption
Ultrasonic

Substituting the fluctuating quantities (3) into eqn (2), expanding and time averaging yields

$$\oiint \bar{\rho}_1 \bar{\alpha} \bar{V}_1 \cdot dA + \frac{\partial}{\partial t} \iiint \bar{\rho}_1 \bar{\alpha}\, dV = \iiint \tilde{\Gamma}_{1m}\, dV$$

$$- \frac{\partial}{\partial t} \iiint \overline{\rho_1' \alpha'}\, dV - \oiint \left(\bar{\rho}_1 \overline{\alpha' V_1'} + \bar{\alpha} \overline{\rho_1' V_1'} + \bar{V}_1 \overline{\rho_1' \alpha'} + \overline{\rho_1' \alpha' V_1'} \right) \cdot dA \quad (5)$$

In Figure 2, a control volume bounded by the channel walls is chosen such that the wall shear stress can be employed and velocity terms other than in the principle flow direction vanish. Employing this control volume allows eqn (5) to be re-written as

$$\frac{\partial}{\partial z} \iint_A \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\, dA dz - \dot{m}'' P_{1m}\, dz + \frac{\partial}{\partial t} \iint_A \bar{\rho}_1 \bar{\alpha}\, dA dz = \iint_A \tilde{\bar{\Gamma}}_{1m}\, dA dz$$

$$- \frac{\partial}{\partial t} \iint_A \overline{\rho_1' \alpha'}\, dA dz - \frac{\partial}{\partial z} \iint_A \left(\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha} \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1' \alpha'} + \overline{\rho_1' \alpha' V_{1z}'} \right) dA dz \quad (6)$$

We define the cross-section-average operator

$$\langle (\quad) \rangle \equiv \frac{1}{A} \iint_A (\quad)\, dA \quad (7)$$

Applying this operator to eqn (6), and lumping the fluctuating terms on the R.H.S. into an effective component 1 mass generation function $\langle \bar{\Gamma}_{1m} \rangle$ for a constant cross-section area A, and no mass flux through the wall yields

$$\frac{\partial}{\partial t} \langle \bar{\rho}_1 \bar{\alpha} \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} \rangle = \langle \bar{\Gamma}_{1m} \rangle \quad (8)$$

Similarly the continuity eqn for component 2 becomes

$$\frac{\partial}{\partial t} \langle \bar{\rho}_2 (1-\bar{\alpha}) \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle = - \langle \bar{\Gamma}_{1m} \rangle = \langle \bar{\Gamma}_{2m} \rangle \quad (9)$$

5.3. CONSERVATION OF MOMENTUM

In a similar manner, the momentum equations for the two phases are

$$\frac{\partial}{\partial t} \langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}^2 \rangle = \langle \bar{\Pi}_{1M} \rangle \tag{10}$$

$$\frac{\partial}{\partial t} \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}^2 \rangle = \langle \bar{\Pi}_{2M} \rangle \tag{11}$$

The momentum equation can be written for the two-phase mixture as

$$\frac{\partial}{\partial t} \langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}^2 + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}^2 \rangle = \left\{ \frac{\partial P}{\partial z} + \frac{\tau_w P_w}{A} + g \langle \bar{\rho} \rangle \cos\theta \right\} \tag{12}$$

where

τ_w = wall shear stress.

$\langle \bar{\rho} \rangle = \langle \bar{\alpha} \bar{\rho}_1 + (1-\bar{\alpha}) \bar{\rho}_2 \rangle$

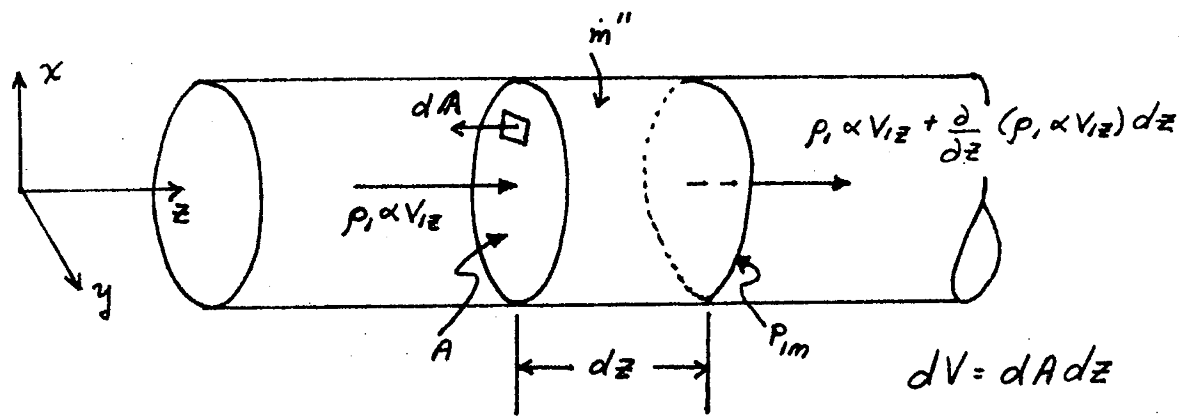


FIGURE 2

5.4. CONSERVATION OF ENERGY

The energy equations for the two components are

$$\frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} \left(\bar{u}_1 + \frac{\overline{\mathbb{V}_1 \cdot \mathbb{V}_1}}{2} + gy \right) > + \frac{\partial}{\partial z} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} \left(\bar{i}_1 + \frac{\overline{\mathbb{V}_1 \cdot \mathbb{V}_1}}{2} + gy \right) > = < \bar{T}_{1u} > \tag{13}$$

$$\frac{\partial}{\partial t} < \bar{\rho}_2 (1-\bar{\alpha}) \left(\bar{u}_2 + \frac{\overline{\mathbb{V}_2 \cdot \mathbb{V}_2}}{2} + gy \right) > + \frac{\partial}{\partial z} < \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \left(\bar{i}_2 + \frac{\overline{\mathbb{V}_2 \cdot \mathbb{V}_2}}{2} + gy \right) > = < \bar{T}_{2u} > \tag{14}$$

The energy equation for the mixture is

$$\frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} \bar{i}_1 + \bar{\rho}_2 (1-\bar{\alpha}) \bar{i}_2 > + \frac{\partial}{\partial z} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} \bar{i}_1 + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \bar{i}_2 > = \frac{\dot{q}'' P_h}{A} + \frac{dw}{A\, dz} \tag{15}$$

5.5. ONE-DIMENSIONAL STEADY HOMOGENEOUS EQUILIBRIUM FLOW

For one-dimensional steady state flows the $\frac{\partial}{\partial t}$ terms vanish in the conservation equations.

The Homogeneous Equilibrium Model (HEM) assumes a mean density for the two-phase mixture, no slip between the phases, and thermodynamic equilibrium. This is also the EVET model (Equal Velocity, Equal Temperature). That is

$$<\bar{\rho}> = <\bar{\alpha}\,\bar{\rho}_1 + (1-\bar{\alpha})\,\bar{\rho}_2> \tag{16}$$

$$<\bar{V}> = <\bar{V}_{1z}> = <\bar{V}_{2z}> \tag{17}$$

$$<i_1> = <i_2> = <i> \tag{18}$$

5.5.1 HEM Continuity Equation

Hence the continuity eqn for the mixture from eqn (8) and (9) becomes

$$\frac{\partial}{\partial z} <\bar{\rho}\,\bar{V}> = 0 \tag{19}$$

or

$$\int\left[\frac{\partial}{\partial z} <\bar{\rho}\,\bar{V}>\right] dz = \text{constant}$$

$$W \equiv A\,\bar{V}\bar{\rho} = \text{constant} \tag{20}$$

5.5.2 HEM Momentum Equation

From eqn (12)

$$\frac{\partial}{\partial z} <\bar{\rho}\,\bar{V}^2> = -\left\{\frac{\partial P}{\partial z} + \frac{\tau_w P_w}{A} + g <\bar{\rho}> \cos\theta\right\}$$

$$\frac{\partial}{\partial z}\left(W\bar{V} + AP\right) = -\tau_w P_w - A\bar{\rho}\, g \cos\theta$$

$$W\frac{d\bar{V}}{dz} = -A\frac{dP}{dz} - P_w \tau_w - A\bar{\rho}\, g \cos\theta \tag{21}$$

5.5.3 HEM Energy Equation

From eqn (15)

$$\frac{\partial}{\partial z} <\bar{\rho}\,\bar{V}\, i> = \frac{\dot{q}'' P_h}{A} + \frac{dw}{A\,dz} \tag{22}$$

or

$$A \bar{V} \bar{\rho} \frac{d}{dz}(i) = \dot{q}'' P_h + \frac{dw}{dz}$$

$$W \frac{d}{dz}\left(u + \frac{\bar{V}^2}{2} + gy\right) = \frac{dQ}{dz} + \frac{dw}{dz} \qquad (23)$$

5.5.4 Further Development and Use of the Momentum Equation

The momentum eqn (21) is often written in terms of the pressure gradient, and its frictional, accelerational, and gravitational components

$$\frac{dP}{dz} = -\frac{P}{A}\tau_\omega - \frac{W}{A}\frac{d\bar{V}}{dz} - \bar{\rho} g \cos\theta$$

$$= \left(\frac{dP}{dz}\right)_F + \left(\frac{dP}{dz}\right)_A + \left(\frac{dP}{dz}\right)_G \qquad (24)$$

5.5.4.1 Frictional Pressure Drop

Expressing the wall shear stress in terms of a two-phase friction factor f_{tp}

$$\tau_\omega = f_{tp} \bar{\rho} \frac{\bar{V}^2}{2} \qquad (25)$$

$$\therefore -\left(\frac{dP}{dz}\right)_F = \frac{P}{A} f_{tp} \bar{\rho} \frac{\bar{V}^2}{2} = \frac{2 f_{tp} \bar{\rho} \bar{V}^2}{D} \qquad (26)$$

or in terms of the mass flux G, specific volumes and mixture quality X

$$-\left(\frac{dP}{dz}\right)_F = \frac{2 f_{tp} G^2}{D}\left(v_f + x v_{fg}\right) \qquad (27)$$

5.5.4.2 Accelerational Pressure Drop

$$-\left(\frac{dP}{dz}\right)_A = \frac{W}{A}\frac{d\bar{V}}{dz} = G \frac{d}{dz}\left(\frac{W}{A\bar{\rho}}\right) \qquad (28)$$

$$\therefore -\left(\frac{dP}{dz}\right)_A = G^2 \frac{d}{dz}\left(\frac{1}{\bar{\rho}}\right) - \left(\frac{G^2}{A\bar{\rho}}\right)\frac{dA}{dz} \qquad (29)$$

Since

$$\frac{d}{dz}\left(\frac{1}{\bar{\rho}}\right) = v_{fg}\frac{dx}{dz} + x\frac{dv_g}{dz} + (1-x)\frac{dv_f}{dz} \tag{30}$$

and in the two-phase region v_f and v_g are functions only of pressure. Hence

$$\frac{d}{dz}\left(\frac{1}{\bar{\rho}}\right) = v_{fg}\frac{dx}{dz} + \frac{dp}{dz}\left[x\frac{dv_g}{dp} + (1-x)\frac{dv_f}{dp}\right] \tag{31}$$

and the acceleration pressure drop becomes

$$-\left(\frac{dP}{dz}\right)_A = G^2\left\{v_{fg}\frac{dx}{dz} + \frac{dp}{dz}\left[x\frac{dv_g}{dp} + (1-x)\frac{dv_f}{dp}\right] - (v_f + x v_{fg})\frac{1}{A}\frac{dA}{dz}\right\} \tag{32}$$

5.5.4.3 Gravitational Pressure Drop

$$-\left(\frac{dP}{dz}\right)_G = \bar{\rho}\, g\cos\theta = \frac{g\cos\theta}{(v_f + x v_{fg})} \tag{33}$$

5.5.4.4 Total Pressure Drop

Substituting eqns. (27), (32), (33) into eqn (24) and re-arranging yields an expression for the total pressure drop.

$$-\left(\frac{dP}{dz}\right) = \frac{\frac{2f_{tp}G^2(v_f + x v_{fg})}{D} + G^2 v_{fg}\frac{dx}{dz} - G^2(v_f + x v_{fg})\frac{1}{A}\frac{dA}{dz} + \frac{g\cos\theta}{(v_f + x v_{fg})}}{1 + G^2\left[x\frac{dv_g}{dp} + (1-x)\frac{dv_f}{dp}\right]} \tag{34}$$

5.6. TWO-PHASE FRICTION FACTOR

The only empirical factor in eqn (34) is the two-phase friction factor. However, a two-phase multiplier ϕ^2_{fo} can be defined such that

$$-\left(\frac{dP}{dz}\right)_F = -\left(\frac{dP}{dz}\right)_{fo} \phi^2_{fo} \tag{35}$$

where the subscript fo indicates the pressure drop evaluated for the case where the total flow is considered to flow as the liquid phase.

i.e.

$$-\left(\frac{dP}{dz}\right)_F = 2f\,\frac{G^2}{D}\,v_f\,\phi^2_{fo} \tag{36}$$

It can be shown that for laminar flow, where $f = \dfrac{64\mu_f A}{DW}$, and $f_{tp} = \dfrac{64\bar{\mu} A}{DW}$ that

$$\phi^2_{fo} = \frac{\rho_f\,\bar{\mu}}{\bar{\rho}\,\mu_f} = \frac{\left[1 + x\left(\frac{\rho_f}{\rho_g} - 1\right)\right]}{\left[1 + x\left(\frac{\mu_f}{\mu_g} - 1\right)\right]} \tag{37}$$

and for turbulent flows where $f = f_{tp}$, that

$$\phi^2_{fo} = 1 + x\left(\frac{\rho_f}{\rho_g} - 1\right) \tag{38}$$

or for smooth pipes where $f = 0.316\ R_f^{-1/4}$, $f_{tp} = 0.316\ R_{tp}^{-1/4}$, that

$$\phi^2_{fo} = \left[1 + x\left(\frac{\mu_f}{\mu_g} - 1\right)\right]^{-1/4}\left[1 + x\left(\frac{\rho_f}{\rho_g} - 1\right)\right] \tag{39}$$

5.7 SUMMARY

The derivation of the cross section averaged form of the mass, energy, and momentum conservation equations in terms of time-averaged variables is presented. The analysis is based on the fact that the variables may fluctuate as turbulent quantities and it is intended to show that the techniques of turbulent flows in general can be applied to two-phase flows.

APPENDIX

1. INTRODUCTION

A single phase flow can be well approximated by a continuum flow. That is, properties such as density, viscosity, velocity, pressure etc., are assumed to vary continuously throughout the fluid. Any discontinuities exist only in the size range of the mean free path of the molecules, which is usually negligible with respect to the size of the flow system.

A two-phase flow may be broadly defined as a flow of two phases (i.e. solid, liquid, or gas) or a flow of two components (i.e. different chemical species) which is piece-wise continuous. That is, the properties such as density, viscosity, velocity etc., are discontinuous step functions over a size range not negligible with respect to the dimensions of the flow system.

Examples of two-phase flows in nature include rain, fog, smog, snow, quicksands, and even the flow of blood. There are many technological examples; among the most important presently being multiphase flows in chemical engineering, and the flow boiling process which occurs in some nuclear power reactors.

Two-phase flows albeit more complicated obey all the basic laws of fluid mechanics. It is intended to show in the following derivation of the conservation equations, that many of the basic techniques of turbulent flows in general can be applied to two-phase flows.

2. THE CONSERVATION OF MASS

2.1 The Integral Formulation

The equation for the conservation of mass in single phase flows is given in differential form by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbb{V} = 0 \qquad (2.1)$$

For turbulent flows, the fluctuating quantities

$$\rho = \bar{\rho} + \rho'$$
$$\mathbb{V} = \bar{\mathbb{V}} + \mathbb{V}'$$

are substituted into the above continuity equation and then time averaged to give

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\bar{\rho}\,\bar{\mathbb{V}} + \overline{\rho' \mathbb{V}'}) = 0 \qquad (2.2)$$

For two-phase flows, however, the differential formulation is not convenient, since it would require complex boundary conditions at the boundaries of each phase region. Thus we adopt an integral formulation in the direction in which the diffusion of mass, momentum, and energy are significant. Therefore, for internal flows the integral formulation in the direction normal to the main flow direction is used. More formally, the cross-section-averaged formulation is used.

The integral formulation is as exact mathematically as the differential formulation. What it does is to reduce the requirement of exact interphase transfer relationships, and since the formulation is integral, these relations need only on the average be correct.

2.2 The Control Volume Approach

For the control volume shown in figure 1, let S_1 be the boundary of a system of particles at time t. After a time interval Δt, the system has moved to the location bounded by S_2. Let X_t be the total mass, momentum, or energy in the system at time t. The volume bounded by S_1 is $V_1 + V_2$, that of S_2 is $V_2 + V_3$.

Fig. 1

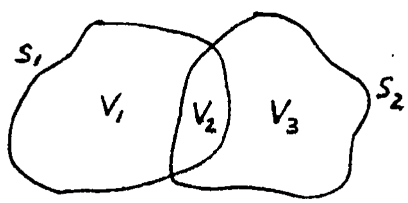
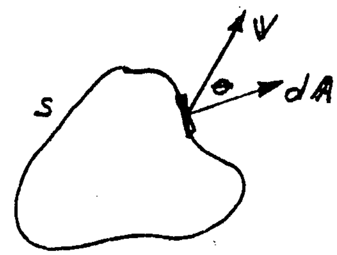


Fig. 2



$$\therefore X_t = X_{V_1 t} + X_{V_2 t}$$

$$X_{t+\Delta t} = X_{V_2\, t+\Delta t} + X_{V_3\, t+\Delta t}$$

The change in X of the system during Δt is

$$X_{t+\Delta t} - X_t = X_{V_2\, t+\Delta t} - X_{V_2 t} + X_{V_3\, t+\Delta t} - X_{V_1 t}$$

$$\frac{\Delta X}{\Delta t} = \frac{\Delta X_{V_2}}{\Delta t} + \frac{X_{V_3\, t+\Delta t} - X_{V_1 t}}{\Delta t} \qquad (2.3)$$

As $\Delta t \to 0$, $V_2 \to V_1$

The first term on the right side becomes the rate of change of X within the control volume.

The second term is the net rate of efflux (the difference between the rate at which $\mathbb{X}$ leaves the control volume and that at which it enters). Let x equal the value of $\mathbb{X}$ per unit mass. In figure 2, $d\mathbb{A}$ is an elemental area on the surface of the control volume, so that the net rate of efflux of $\mathbb{X}$ across $d\mathbb{A}$ is given by $x(\rho\mathbb{V}\cdot d\mathbb{A})$.

Eqn (2.3) can be written as

$$\frac{dX}{dt} = \left.\frac{dX}{dt}\right|_V + \oint\!\!\oint x(\rho\mathbb{V}\cdot d\mathbb{A})$$

Rate of Generation	Rate of Increase in the C.V.	Net Efflux Integrated Over the C.S.

or

$$\frac{dX}{dt} = \frac{\partial}{\partial t}\iiint x\rho\, dV + \oint\!\!\oint x(\rho\mathbb{V}\cdot d\mathbb{A}) \qquad (2.4)$$

Eqn. (2.4) is the general conservation equation for a control volume.

2.3 The Continuity Equation (for Component 1)

From eqn (2.4) the mass conservation equation for component 1 becomes

$$\oint\!\!\oint \tilde{\rho}_1 \mathbb{V}_1\cdot d\mathbb{A} + \frac{\partial}{\partial t}\iiint \tilde{\rho}_1\, dV = \iiint \tilde{\Gamma}_{1m}\, dV \qquad (2.5)$$

Net Rate of Efflux From C.V.	Rate of Increase of Stored Mass in C.V.	Mass Generation in C.V.

where we define

$\tilde{\rho}_1$ = instantaneous density of component 1
$\mathbb{V}_1$ = velocity of component 1 (a vector)
$d\mathbb{A}$ = area vector of magnitude dA and direction normal and outwards from dA
t = time
dV = element of volume
$\tilde{\Gamma}_{1m}$ = volumetric rate of creation of component 1, or the component 1 mass generation function

2.3.a Void Fraction

We define a point function α such that $\alpha=1$ when component 1 occupies the point and $\alpha=0$ when component 1 is absent at the point. Thus $\tilde{\rho}_1=\rho_1\alpha$ where ρ_1 is the continuous density of component 1

Thus by definition

$$\frac{1}{V}\iiint \alpha \, dV = \frac{V_1}{V} \tag{2.6}$$

That is, the instantaneous average of α over the volume represents the fraction of the volume occupied by component 1. Component 1 is the lighter component (steam in a steam-water flow, air in an air-water flow) and the space average value of α is called the instantaneous void fraction or the vapour volume fraction.

2.3.b Time Averaging

As in turbulent single phase flows, two-phase flows exhibit fluctuations, and thus instantaneous values alone are of little use because they tell us nothing about the magnitude of the fluctuations.

Thus the time-averaging operator is defined by

$$(\overline{}) \equiv \frac{1}{\Delta t}\int_{\Delta t} (\quad) \, dt \tag{2.7}$$

As for time averaging in turbulent flows, it must be noted that Δt is a finite time interval. It must be large compared to the time scale T_1 of the fluctuations and on the other hand be small compared with the period T_2 of any slow variations in the flow field that we do not wish to regard as belonging to the fluctuations.

Flows in which the characteristic time scale T_2 of the gross changes is very much larger than the characteristic time scale T_1 of the detailed fluctuations in the various quantities, may be termed slow transients.

2.3.c Derivation

$$\because \oiint \tilde{\rho}_1 \mathbb{V}_1 \cdot d\mathbb{A} + \frac{\partial}{\partial t}\iiint \tilde{\rho}_1 \, dV = \iiint \tilde{\Gamma}_{1m} \, dV \tag{2.5}$$

$$\because \tilde{\rho}_1 = \rho_1 \alpha$$

$$\therefore \oiint \rho_1 \alpha \mathbb{V}_1 \cdot d\mathbb{A} + \frac{\partial}{\partial t}\iiint \rho_1 \alpha \, dV = \iiint \tilde{\Gamma}_{1m} \, dV \tag{2.8}$$

Considering fluctuations of quantities

$$\begin{aligned} \rho_1 &= \bar{\rho}_1 + \rho_1' \qquad & \mathbb{V}_1 &= \overline{\mathbb{V}}_1 + \mathbb{V}_1' \\ \alpha &= \bar{\alpha} + \alpha' \qquad & \tilde{\Gamma}_{1m} &= \bar{\tilde{\Gamma}}_{1m} + \tilde{\Gamma}_{1m}' \end{aligned} \tag{2.9}$$

Substituting eqns (2.9) into eqn (2.8)

$$\oiint (\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')(\bar{V}_1 + V_1') \cdot dA + \frac{\partial}{\partial t}\iiint (\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')\, dV = \iiint (\bar{\tilde{T}}_{1m} + \tilde{T}_{1m}')\, dV$$

Expanding the above equation and applying the time-averaging operator

$$\oiint (\overline{\bar{\rho}_1 \bar{\alpha} \bar{V}_1} + \overline{\bar{\rho}_1 \alpha' \bar{V}_1} + \overline{\rho_1' \bar{\alpha} \bar{V}_1} + \overline{\rho_1' \alpha' \bar{V}_1} + \overline{\bar{\rho}_1 \bar{\alpha} V_1'} + \overline{\bar{\rho}_1 \alpha' V_1'} + \overline{\rho_1' \bar{\alpha} V_1'} + \overline{\rho_1' \alpha' V_1'}) \cdot dA$$

$$+ \frac{\partial}{\partial t}\iiint (\overline{\bar{\rho}_1 \bar{\alpha}} + \overline{\bar{\rho}_1 \alpha'} + \overline{\rho_1' \bar{\alpha}} + \overline{\rho_1' \alpha'})\, dV$$

$$= \iiint (\overline{\bar{\tilde{T}}_{1m}} + \overline{\tilde{T}_{1m}'})\, dV$$

The resulting equation is

$$\underbrace{\oiint \bar{\rho}_1 \bar{\alpha} \bar{V}_1 \cdot dA}_{I} + \underbrace{\frac{\partial}{\partial t}\iiint \bar{\rho}_1 \bar{\alpha}\, dV}_{II} = \underbrace{\iiint \bar{\tilde{T}}_{1m}\, dV}_{III}$$

$$\underbrace{- \frac{\partial}{\partial t}\iiint \overline{\rho_1' \alpha'}\, dV}_{IV}$$

$$\underbrace{- \oiint (\bar{\rho}_1 \overline{\alpha' V_1'} + \bar{\alpha}\, \overline{\rho_1' V_1'} + \bar{V}_1 \overline{\rho_1' \alpha'} + \overline{\rho_1' \alpha' V_1'}) \cdot dA}_{V} \qquad (2.10)$$

Note that the time averaging of the products of quantities follow the general rules of turbulent flows.

The terms in eqn (2.10) are:

I The net rate of efflux from the C.V. of the average mass flow rate of component 1 over Δt, by the average density, void, and velocity.

II The average rate of increase of stored mass of component 1 in the C.V. due to the rates of increase of the average density and void fraction.

III The average component 1 mass generation in the C.V.

IV The rate of increase of stored mass of component 1 in the C.V., due to the rates of increase of the fluctuations in density and void.

V The net rate of efflux from the C.V. by the fluctuations in density, void, and velocity.

2.3.d The Control Volume

The choice of a suitable control volume is essential to yield a tractable solution. We may choose a control volume the boundary of which may or may not be fully made up of a fluid surface.

For an internal flow in a channel we can choose either a small elemental volume within the channel, or we may select the channel inside surface or walls as part of the control volume boundary. If we choose the former control volume; the integrals across the control volume and control surface will be impossible to solve since we know nothing about the boundary conditions such as the shear stress, and there will be net momentum transfer across all of its surfaces.

By choosing a control volume bounded by the channel walls we know something about the boundary conditions. We may use a suitable wall shear stress correlation and assume suitable distribution functions. Also by choosing the fluid boundaries of the control volume normal to the principal flow direction, the scalar product of the velocity vector $\mathbb{V}_1 = (\bar{V}_{1x} + V'_{1x})\vec{i} + (\bar{V}_{1y} + V'_{1y})\vec{j} + (\bar{V}_{1z} + V'_{1z})\vec{k}$ with the elemental area vector $d\mathbf{A}$, causes the velocity terms in the x and y directions to vanish. That is:

$$\oiint \bar{\rho}_1 \bar{\alpha} \bar{\mathbb{V}}_1 \cdot d\mathbf{A} = \oiint \bar{\rho}_1 \bar{\alpha} (\bar{V}_{1x} \vec{i} + \bar{V}_{1y} \vec{j} + \bar{V}_{1z} \vec{k}) \cdot dA\, \vec{k}$$

$$\because \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0$$

$$\vec{k} \cdot \vec{k} = 1$$

$$\therefore \oiint \bar{\rho}_1 \bar{\alpha} \bar{\mathbb{V}}_1 \cdot d\mathbf{A} = \oiint (\bar{\rho}_1 \bar{\alpha} \bar{V}_{1z})\, dA$$

There is also no net change of momentum through the control volume in the direction normal to the principal flow direction (e.g. radially) because of the solid wall boundaries. Thus the net change in momentum occurs in the principal flow direction only. That is:

$$\frac{\partial}{\partial t} \iiint \bar{\rho}_1 \bar{\alpha} \bar{\mathbb{V}}_1\, dV = \frac{\partial}{\partial t} \iiint \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\, dV$$

Considering a control volume as shown in figure 3, representing say, a section of pipe.

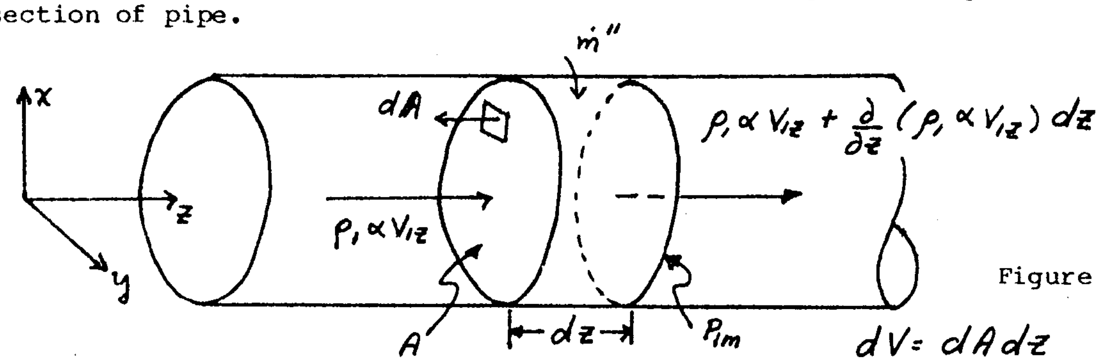


Figure 3

where A = constant cross-section area through dz
P_{1m} = portion of the perimeter through which component 1 is transferred
$\dot{m}''$ = rate of mass per unit area of component 1 transferred through the side of the control volume

Now in eqn. (2.10) term I becomes

$$\oint\!\!\oint \bar{\rho}_1 \bar{\alpha} \bar{V}_1 \cdot dA = \left[\iint_A \bar{\rho}_1 \bar{\alpha} V_{1z}\, dA + \iint_A \frac{\partial}{\partial z}(\bar{\rho}_1 \bar{\alpha} \bar{V}_{1z})\, dz\, dA\right] - \iint_A \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\, dA$$

$$- \dot{m}'' P_{1m}\, dz$$

$$= \frac{\partial}{\partial z}\iint_A \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\, dA\, dz - \dot{m}'' P_{1m}\, dz$$

Term II becomes

$$\frac{\partial}{\partial t}\iiint \bar{\rho}_1 \bar{\alpha}\, dV = \frac{\partial}{\partial t}\iint_A \int_{dz} \bar{\rho}_1 \bar{\alpha}\, dA\, dz$$

$$= \frac{\partial}{\partial t}\iint_A \bar{\rho}_1 \bar{\alpha} \left(\int_0^{dz} dz\right) dA$$

$$= \frac{\partial}{\partial t}\iint_A \bar{\rho}_1 \bar{\alpha}\, dA\, dz$$

Term III becomes

$$\iiint \bar{\tilde{T}}_{1m}\, dV = \iint_A \int_{dz} \bar{\tilde{T}}_{1m}\, dA\, dz = \iint_A \bar{\tilde{T}}_{1m}\, dA\, dz$$

Term IV becomes

$$-\frac{\partial}{\partial t}\iiint \overline{\rho_1'\alpha'}\, dV = -\frac{\partial}{\partial t}\iint_A \overline{\rho_1'\alpha'}\, dA\, dz$$

Term V becomes

$$-\oint\!\!\!\oint (\bar{\rho}_1 \overline{\alpha' \mathbf{V}_1'} + \bar{\alpha}\, \overline{\rho_1' \mathbf{V}_1'} + \bar{\mathbf{V}}_1 \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha'\mathbf{V}_1'}) \cdot d\mathbf{A}$$
$$= \Big\{ \iint_A (\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha}\, \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_{1z}'})\, dA$$
$$+ \iint_A \frac{\partial}{\partial z} (\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha}\, \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_{1z}'})\, dA\, dz$$
$$- \iint_A (\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha}\, \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_{1z}'})\, dA \Big\}$$
$$= -\frac{\partial}{\partial z}\iint_A (\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha}\, \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_{1z}'})\, dA\, dz$$

and thus eqn. (2.10) becomes

$$\frac{\partial}{\partial z}\iint_A \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\, dA\, dz - \dot{m}'' P_{1m}\, dz + \frac{\partial}{\partial t}\iint_A \bar{\rho}_1 \bar{\alpha}\, dA\, dz = \iint_A \bar{\tilde{\Gamma}}_{1m}\, dA\, dz$$
$$- \frac{\partial}{\partial t}\iint_A \overline{\rho_1'\alpha'}\, dA\, dz$$
$$- \frac{\partial}{\partial z}\iint_A (\bar{\rho}_1 \overline{\alpha' V_{1z}'} + \bar{\alpha}\, \overline{\rho_1' V_{1z}'} + \bar{V}_{1z} \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_{1z}'})\, dA\, dz \qquad (2.11)$$

We define the cross-section-average operator

$$\langle (\ \ ) \rangle \equiv \frac{1}{A}\iint_A (\ \ )\, dA$$

and applying $\langle (\ ) \rangle$ to eqn. (2.11)

$$\frac{\partial}{\partial z}[A \langle \bar{\rho}_1 \bar{\alpha} V_{1z} \rangle]\, dz - \dot{m}'' P_{1m}\, dz + \frac{\partial}{\partial t}[A \langle \bar{\rho}_1 \bar{\alpha} \rangle]\, dz = A \langle \bar{\tilde{\Gamma}}_{1m} \rangle\, dz$$
$$- \frac{\partial}{\partial t}[A \langle \overline{\rho_1'\alpha'} \rangle]\, dz$$
$$- \frac{\partial}{\partial z}\Big[A \Big\{ \langle \bar{\rho}_1 \overline{\alpha' V_{1z}'} \rangle + \langle \bar{\alpha}\, \overline{\rho_1' V_{1z}'} \rangle + \langle \bar{V}_{1z} \overline{\rho_1'\alpha'} \rangle + \langle \overline{\rho_1'\alpha' V_{1z}'} \rangle \Big\}\Big] dz \qquad (2.12)$$

Dividing eqn. (2.12) by dz gives

$$\frac{\partial}{\partial t}\,[A\langle\bar{\rho}_1\bar{\alpha}\rangle] + \frac{\partial}{\partial z}\,[A\langle\bar{\rho}_1\bar{\alpha}\bar{V}_{1z}\rangle] - \dot{m}''P_{1m} = A\langle\bar{\bar{\pi}}_{1m}\rangle$$

$$-\frac{\partial}{\partial t}\,[A\langle\overline{\rho_1'\alpha'}\rangle]$$

$$-\frac{\partial}{\partial z}\,\left[A\left\{\langle\bar{\rho}_1\overline{\alpha' V_{1z}'}\rangle + \langle\bar{\alpha}\overline{\rho_1' V_{1z}'}\rangle + \langle\bar{V}_{1z}\overline{\rho_1'\alpha'}\rangle + \langle\overline{\rho_1'\alpha' V_{1z}'}\rangle\right\}\right] \quad (2.13)$$

2.3.e Simplifications

The right hand side of eqn. (2.13) contains the fluctuating terms which are difficult to evaluate. Thus we lump these parameters in an effective component 1 mass generation function defined as

$$\langle\bar{\pi}_{1m}\rangle \equiv \langle\bar{\bar{\pi}}_{1m}\rangle - \frac{1}{A}\frac{\partial}{\partial t}\,[A\langle\overline{\rho_1'\alpha'}\rangle]$$

$$-\frac{1}{A}\frac{\partial}{\partial z}\left[A\left\{\langle\bar{\rho}_1\overline{\alpha' V_{1z}'}\rangle + \langle\bar{\alpha}\overline{\rho_1' V_{1z}'}\rangle + \langle\bar{V}_{1z}\overline{\rho_1'\alpha'}\rangle + \langle\overline{\rho_1'\alpha' V_{1z}'}\rangle\right\}\right]$$

and eqn. (2.13) may be written as

$$\frac{\partial}{\partial t}\,[A\langle\bar{\rho}_1\bar{\alpha}\rangle] + \frac{\partial}{\partial z}\,[A\langle\bar{\rho}_1\bar{\alpha}\bar{V}_{1z}\rangle] - \dot{m}''P_{1m} = A\langle\bar{\pi}_{1m}\rangle \quad (2.14)$$

When the cross-section area A is constant and there is no mass flux through the wall (i.e. $\dot{m}''=0$) eqn. (2.14) becomes

$$\frac{\partial}{\partial t}\,\langle\bar{\rho}_1\bar{\alpha}\rangle + \frac{\partial}{\partial z}\,\langle\bar{\rho}_1\bar{\alpha}\bar{V}_{1z}\rangle = \langle\bar{\pi}_{1m}\rangle \quad (2.15)$$

2.4 The Continuity Equation (Component 2)

Since α is defined as the volume of that portion of V which is occupied by component 1

$$\therefore\ \frac{1}{V}\iiint \alpha\, dV = \frac{V_1}{V}$$

thus the portion of V occupied by component 2 is $(1-\alpha)$

$$\therefore\ \frac{1}{V}\iiint (1-\alpha)\, dV = \frac{V_2}{V}$$

Thus the mass conservation equation of component 2 is similar to eqn. (2.8)

$$\oiint \rho_2 (1-\alpha) \mathbb{V}_2 \cdot dA + \frac{\partial}{\partial t} \iiint \rho_2 (1-\alpha)\, dV = \iiint \tilde{\Gamma}_{2m}\, dV \qquad (2.16)$$

Considering fluctuations of quantities

$$\begin{aligned} \rho_2 &= \bar{\rho}_2 + \rho_2{}' \\ \mathbb{V}_2 &= \bar{\mathbb{V}}_2 + \mathbb{V}_2{}' \\ \tilde{\Gamma}_{2m} &= \bar{\tilde{\Gamma}}_{2m} + \tilde{\Gamma}_{2m}{}' \\ (1-\alpha) &= (1-\bar{\alpha}) + (1-\alpha') \end{aligned} \qquad (2.17)$$

Substituting eqns. (2.17) and time-averaging yields;

$$\begin{aligned} &\oiint \bar{\rho}_2 (1-\bar{\alpha}) \bar{\mathbb{V}}_2 \cdot dA + \frac{\partial}{\partial t} \iiint \bar{\rho}_2 (1-\bar{\alpha})\, dV = \iiint \bar{\tilde{\Gamma}}_{2m}\, dV \\ &- \frac{\partial}{\partial t} \iiint \overline{\rho_2{}'(1-\alpha')}\, dV \\ &- \oiint \left\{ \bar{\rho}_2 \overline{(1-\alpha')\mathbb{V}_2{}'} + (1-\bar{\alpha}) \overline{\rho_2{}'\mathbb{V}_2{}'} + \bar{\mathbb{V}}_2 \overline{(1-\alpha')\rho_2{}'} + \overline{\rho_2{}'(1-\alpha')\mathbb{V}_2{}'} \right) \cdot dA \end{aligned} \qquad (2.18)$$

Using the same control volume as previously, and incorporating a term $\dot{m}_2'' P_{2m}$ which represents the mass transfer of component 2 through the edge of the control volume, we obtain

$$\begin{aligned} &\frac{\partial}{\partial z} \iint_A \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}\, dA dz - \dot{m}_2'' P_{2m}\, dz + \frac{\partial}{\partial t} \iint_A \bar{\rho}_2 (1-\bar{\alpha})\, dA dz = \iint_A \bar{\tilde{\Gamma}}_{2m}\, dA dz \\ &- \frac{\partial}{\partial t} \iint_A \overline{\rho_2{}'(1-\alpha')}\, dA dz \\ &- \frac{\partial}{\partial z} \iint_A \left\{ \bar{\rho}_2 \overline{(1-\alpha')V_{2z}'} + (1-\bar{\alpha}) \overline{\rho_2{}'V_{2z}'} + \bar{V}_{2z} \overline{(1-\alpha')\rho_2{}'} + \overline{\rho_2{}'(1-\alpha')V_{2z}'} \right\} dA dz \end{aligned} \qquad (2.19)$$

Applying the cross-section-average operator $\langle\ \rangle$

$$\begin{aligned} &\frac{\partial}{\partial z} \left[A \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle \right] - \dot{m}_2'' P_{2m} + \frac{\partial}{\partial t} \left[A \langle \bar{\rho}_2 (1-\bar{\alpha}) \rangle \right] = A \langle \bar{\tilde{\Gamma}}_{2m} \rangle \\ &- \frac{\partial}{\partial t} \left[A \langle \overline{\rho_2{}'(1-\alpha')} \rangle \right] \\ &- \frac{\partial}{\partial z} \left[A \left\{ \langle \bar{\rho}_2 \overline{(1-\alpha')V_{2z}'} \rangle + \langle (1-\bar{\alpha}) \overline{\rho_2{}'V_{2z}'} \rangle + \langle \bar{V}_{2z} \overline{(1-\alpha')\rho_2{}'} \rangle + \langle \overline{\rho_2{}'(1-\alpha')V_{2z}'} \rangle \right\} \right] \end{aligned} \qquad (2.20)$$

Defining an effective component 2 mass generation function as

$$\langle \bar{\Gamma}_{2m} \rangle \equiv \langle \tilde{\bar{\Gamma}}_{2m} \rangle - \frac{1}{A}\frac{\partial}{\partial t}\left[A \langle \overline{\rho_2'(1-\alpha')} \rangle \right]$$

$$- \frac{1}{A}\frac{\partial}{\partial z}\left[A \left\{ \langle \bar{\rho}_2 \overline{(1-\alpha')V_{2z}'} \rangle + \langle (1-\bar{\alpha}) \overline{\rho_2' V_{2z}'} \rangle + \langle \bar{V}_{2z} \overline{(1-\alpha')\rho_2'} \rangle + \langle \rho_2'(1-\alpha') V_{2z}' \rangle \right\} \right] \tag{2.21}$$

we obtain

$$\frac{\partial}{\partial t}\left[A \langle \bar{\rho}_2 (1-\bar{\alpha}) \rangle \right] + \frac{\partial}{\partial z}\left[A \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle \right] - \dot{m}_2'' P_{2m} = A \langle \bar{\Gamma}_{2m} \rangle \tag{2.22}$$

When the cross-section area A is constant and there is no mass flux through the wall ($\dot{m}_2'' = 0$), eqn. (2.22) becomes

$$\frac{\partial}{\partial t} \langle \bar{\rho}_2 (1-\bar{\alpha}) \rangle + \frac{\partial}{\partial z} \langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \rangle = \langle \bar{\Gamma}_{2m} \rangle \tag{2.23}$$

3. THE CONSERVATION OF MOMENTUM

For an inertial reference, Newton's Second Law of Motion as applied to a fixed mass M is given by

$$\Sigma \mathbb{F} = \frac{M}{g_c} \frac{d\mathbb{V}}{dt} \qquad \Sigma \mathbb{F} = \frac{1}{g_c} \frac{d\mathbb{P}}{dt}$$

where $\mathbb{P}$ is the linear momentum of the system, and the sum of the forces $\Sigma \mathbb{F}$ includes body forces and surface tractions. In eqn. (2.4) letting $X = \mathbb{P}$ and $x = \frac{\mathbb{P}}{M} = \mathbb{V}$, the momentum equation for a control volume may be written for component 1 as

$$\oiint \mathbb{V}_1 (\tilde{\rho}_1 \mathbb{V}_1 \cdot d\mathbb{A}) + \frac{\partial}{\partial t} \iiint \mathbb{V}_1 \tilde{\rho}_1 \, dV = \frac{d\mathbb{P}}{dt} \qquad (3.1)$$

$$\because \tilde{\rho}_1 = \rho_1 \alpha \;, \qquad \Sigma \mathbb{F} = \frac{1}{g_c} \frac{d\mathbb{P}}{dt}$$

$$\oiint \mathbb{V}_1 \frac{(\rho_1 \alpha \mathbb{V}_1 \cdot d\mathbb{A})}{g_c} + \frac{\partial}{\partial t} \iiint \frac{\mathbb{V}_1 \rho_1 \alpha}{g_c} \, dV = \frac{1}{g_c} \frac{d\mathbb{P}}{dt} = \Sigma \mathbb{F} \qquad (3.2)$$

Defining a component 1 momentum generation term $\tilde{\mathcal{T}}_{1M}$ as

$$\iiint \tilde{\mathcal{T}}_{1M} \, dV = \frac{1}{g_c} \frac{d\mathbb{P}}{dt} \qquad (3.3)$$

$$\therefore \oiint \mathbb{V}_1 \frac{(\rho_1 \alpha \mathbb{V}_1 \cdot d\mathbb{A})}{g_c} + \frac{\partial}{\partial t} \iiint \frac{\mathbb{V}_1 \rho_1 \alpha}{g_c} \, dV = \iiint \tilde{\mathcal{T}}_{1M} \, dV \qquad (3.4)$$

Considering fluctuations of quantities

$$\begin{aligned} \rho_1 &= \bar{\rho}_1 + \rho_1' \\ \alpha &= \bar{\alpha} + \alpha' \\ \mathbb{V}_1 &= \bar{\mathbb{V}}_1 + \mathbb{V}_1' \\ \tilde{\mathcal{T}}_{1M} &= \bar{\tilde{\mathcal{T}}}_{1M} + \tilde{\mathcal{T}}_{1M}' \end{aligned} \qquad (3.5)$$

Substituting eqns. (3.5) into eqn. (3.4)

$$\oiint (\bar{\mathbb{V}}_1 + \mathbb{V}_1') \left[\frac{(\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')(\bar{\mathbb{V}}_1 + \mathbb{V}_1') \cdot d\mathbb{A}}{g_c} \right] + \frac{\partial}{\partial t} \iiint \frac{(\bar{\mathbb{V}}_1 + \mathbb{V}_1')(\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')}{g_c} \, dV$$

$$= \iiint (\bar{\tilde{\mathcal{T}}}_{1M} + \tilde{\mathcal{T}}_{1M}') \, dV \qquad (3.6)$$

Thus the momentum conservation equation for component 1 is given as (when g_c incorporated correctly)

$$\frac{\partial}{\partial t}\left[A\langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\rangle\right] + \frac{\partial}{\partial z}\left[A\langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}^2\rangle\right] = A\langle \bar{\tau}_{1M}\rangle \tag{3.7}$$

Similarly, the momentum equation for component 2 becomes

$$\frac{\partial}{\partial t}\left[A\langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}\rangle\right] + \frac{\partial}{\partial z}\left[A\langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}^2\rangle\right] = A\langle \bar{\tau}_{2M}\rangle \tag{3.8}$$

When the cross-sectional area is constant, eqns. (3.7) and (3.8) become

$$\frac{\partial}{\partial t}\langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}\rangle + \frac{\partial}{\partial z}\langle \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}^2\rangle = \langle \bar{\tau}_{1M}\rangle \tag{3.9}$$

$$\frac{\partial}{\partial t}\langle \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}\rangle + \frac{\partial}{\partial z}\langle \bar{\rho}_2 (1-\bar{\alpha} \bar{V}_{2z}^2\rangle = \langle \bar{\tau}_{2M}\rangle \tag{3.10}$$

4. THE ENERGY CONSERVATION EQUATION

The First Law of Thermodynamics for a system consisting of a fixed mass of particles is given by

$$dE = đQ - đW$$

where Q = heat crossing system boundaries

W = mechanical, electrical, magnetic, surface tension etc. types of work done by the system

E = $U + KE + PE$ (internal + kinetic + potential) energy

Thus from eqn. (2.4) the energy equation for a control volume may be written for component 1 as

$$\oiint e\,(\tilde{\rho}_1 \mathbb{V}_1 \cdot d\mathbb{A}) + \frac{\partial}{\partial t}\iiint e\,\tilde{\rho}_1\,dV = \frac{dE}{dt} = \frac{đQ}{dt} - \frac{đW}{dt} \tag{4.1}$$

where e equals the total energy E per unit mass

$$\because \quad E = U + KE + PE$$

$$\therefore \quad e = u_1 + \frac{1}{2}\frac{(\mathbb{V}_1 \cdot \mathbb{V}_1)}{g_c} + \frac{g\,y}{g_c}$$

where u_1 is the internal energy per unit mass of component 1.

Substituting, eqn. (4.1) becomes

$$\oiint \left(u_1 + \frac{(\mathbb{V}_1 \cdot \mathbb{V}_1)}{2g_c} + \frac{g\,y}{g_c}\right)(\tilde{\rho}_1 \mathbb{V}_1 \cdot d\mathbb{A}) + \frac{\partial}{\partial t}\iiint \tilde{\rho}_1\left(u_1 + \frac{(\mathbb{V}_1 \cdot \mathbb{V}_1)}{2g_c} + \frac{g\,y}{g_c}\right)dV = \frac{d}{dt}(Q - W) \tag{4.2}$$

It is convenient to divide the work W into the flow work necessary to push the mass across the boundaries of the control volume; and all the other work crossing the control surface. Consider a mass of component 1 ΔM of volume ΔV which flows across the control surface. The work done by the system to push the mass across the boundary against the pressure acting externally on the boundary is $P\Delta V$. Since the mass density of component 1 is $\rho_1 = \frac{\Delta M}{\Delta V}$, thus the flow work done by the system per unit mass of component 1 is $\frac{P}{\rho_1}$.

Eqn. (4.2) may now be written as

$$\oiint \left(u_1 + \frac{P_1}{\rho} + \frac{\mathbb{V}_1 \cdot \mathbb{V}_1}{2 g_c} + \frac{g y}{g_c}\right)(\tilde{\rho}_1 \mathbb{V}_1 \cdot d\mathbb{A}) + \frac{\partial}{\partial t} \iiint \tilde{\rho}_1 \left(u_1 + \frac{\mathbb{V}_1 \cdot \mathbb{V}_1}{2 g_c} + \frac{g y}{g_c}\right) dV \equiv \frac{d}{dt}(Q - w) \tag{4.3}$$

where now w includes all work except for flow work

Defining a component 1 energy generation function $\tilde{T}_{1u}$ as

$$\iiint \tilde{T}_{1u} \, dV \equiv \frac{d}{dt}(Q - w)$$

and also since the thermodynamic property enthalpy $i = u + \dfrac{P}{\rho}$

$$\oiint \rho_1 \alpha \mathbb{V}_1 \left(i_1 + \frac{\mathbb{V}_1 \cdot \mathbb{V}_1}{2 g_c} + \frac{g y}{g_c}\right) \cdot d\mathbb{A} + \frac{\partial}{\partial t} \iiint \rho_1 \alpha \left(u_1 + \frac{\mathbb{V}_1 \cdot \mathbb{V}_1}{2 g_c} + \frac{g y}{g_c}\right) dV = \iiint \tilde{T}_{1u} \, dV \tag{4.4}$$

Considering fluctuations of quantities

$$\begin{aligned} \rho_1 &= \bar{\rho}_1 + \rho_1' \\ \alpha &= \bar{\alpha} + \alpha' \\ \mathbb{V}_1 &= \bar{\mathbb{V}}_1 + \mathbb{V}_1' \\ i_1 &= \bar{i}_1 + i_1' \\ u_1 &= \bar{u}_1 + u_1' \\ \tilde{T}_{1u} &= \bar{\tilde{T}}_{1u} + \tilde{T}_{1u}' \end{aligned} \tag{4.5}$$

Eqn. (4.4) becomes

$$\oiint (\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')(\bar{\mathbb{V}}_1 + \mathbb{V}_1')\left[(\bar{i}_1 + i_1') + \frac{(\bar{\mathbb{V}}_1 + \mathbb{V}_1') \cdot (\bar{\mathbb{V}}_1 + \mathbb{V}_1')}{2 g_c} + \frac{g y}{g_c}\right] \cdot d\mathbb{A}$$

$$+ \frac{\partial}{\partial t} \iiint (\bar{\rho}_1 + \rho_1')(\bar{\alpha} + \alpha')\left[(\bar{u}_1 + u_1') + \frac{(\bar{\mathbb{V}}_1 + \mathbb{V}_1') \cdot (\bar{\mathbb{V}}_1 + \mathbb{V}_1')}{2 g_c} + \frac{g y}{g_c}\right] dV \tag{4.6}$$

$$= \iiint (\bar{\tilde{T}}_{1u} + \tilde{T}_{1u}') \, dV$$

Applying the cross-section-average operator $< >$, the cross-section-average form of the energy equation for component 1 is obtained (incorporating g_c correctly) as

$$\frac{\partial}{\partial t}\left[A<\bar{\rho}_1\bar{\alpha}\left(\bar{u}_1+\frac{\overline{\mathbb{V}_1\cdot\mathbb{V}_1}}{2}+gy\right)>\right]+\frac{\partial}{\partial z}\left[A<\bar{\rho}_1\bar{\alpha}\bar{V}_{1z}\left(\bar{i}_1+\frac{\overline{\mathbb{V}_1\cdot\mathbb{V}_1}}{2}+gy\right)>\right] = A<\bar{T}_{1u}> \tag{4.7}$$

Similarly, the energy equation for component 2 becomes

$$\frac{\partial}{\partial t}\left[A<\bar{\rho}_2(1-\bar{\alpha})\left(\bar{u}_2+\frac{\overline{\mathbb{V}_2\cdot\mathbb{V}_2}}{2}+gy\right)>\right]+\frac{\partial}{\partial z}\left[A<\bar{\rho}_2(1-\bar{\alpha})\bar{V}_{2z}\left(\bar{i}_2+\frac{\overline{\mathbb{V}_2\cdot\mathbb{V}_2}}{2}+gy\right)>\right] = A<\bar{T}_{2u}> \tag{4.8}$$

When the cross-sectional area is constant, eqns. (4.7) and (4.8) become

$$\frac{\partial}{\partial t}<\bar{\rho}_1\bar{\alpha}\left(\bar{u}_1+\frac{\overline{\mathbb{V}_1\cdot\mathbb{V}_1}}{2}+gy\right)>+\frac{\partial}{\partial z}<\bar{\rho}_1\bar{\alpha}\bar{V}_{1z}\left(\bar{i}_1+\frac{\overline{\mathbb{V}_1\cdot\mathbb{V}_1}}{2}+gy\right)> = <\bar{T}_{1u}> \tag{4.9}$$

$$\frac{\partial}{\partial t}<\bar{\rho}_2(1-\bar{\alpha})\left(\bar{u}_2+\frac{\overline{\mathbb{V}_2\cdot\mathbb{V}_2}}{2}+gy\right)>+\frac{\partial}{\partial z}<\bar{\rho}_2(1-\bar{\alpha})\bar{V}_{2z}\left(\bar{i}_2+\frac{\overline{\mathbb{V}_2\cdot\mathbb{V}_2}}{2}+gy\right)> = <\bar{T}_{2u}> \tag{4.10}$$

5. THE FLUCTUATING TERMS IN THE CONSERVATION EQUATIONS - A CONSIDERATION OF THE EFFECTIVE GENERATION FUNCTION

5.1 The Continuity Equation

In eqn. (2.10) the fluctuating term IV

$$\frac{\partial}{\partial t}\iiint \overline{\rho_1'\alpha'}\, dV$$

has already been described as the rate of increase of stored mass of component 1 in the C.V. due to the rates of increase of the fluctuations in density and void.

Also, term V in eqn. (2.10)

$$\oiint (\bar{\rho}_1 \overline{\alpha' V_1'} + \bar{\alpha}\, \overline{\rho_1' V_1'} + \bar{V}_1 \overline{\rho_1'\alpha'} + \overline{\rho_1'\alpha' V_1'}) \cdot dA$$

has been described as the net rate of efflux from the C.V. of the mass due to the fluctuations in density, void, and velocity.

After applying the control volume and cross-section-averaging operation, the above terms were lumped into the effective component 1 mass generation function $\langle \bar{\Gamma}_{1m} \rangle$ defined as

$$\langle \bar{\Gamma}_{1m} \rangle \equiv \langle \bar{\bar{\Gamma}}_{1m} \rangle - \frac{1}{A}\frac{\partial}{\partial t}\left[A \langle \overline{\rho_1'\alpha'} \rangle\right]$$

$$-\frac{1}{A}\frac{\partial}{\partial z}\left[A\left\{\langle \bar{\rho}_1 \overline{\alpha' V_{1z}'} \rangle + \langle \bar{\alpha}\, \overline{\rho_1' V_{1z}'} \rangle + \langle \bar{V}_{1z} \overline{\rho_1'\alpha'} \rangle + \langle \overline{\rho_1'\alpha' V_{1z}'} \rangle\right\}\right]$$

The term $\langle \overline{\rho_1'\alpha'} \rangle$ could be considered a compressability term, the ρ_1' being fluctuations in the vapour density and α' being fluctuations in the void. It is obvious that fluctuations in either or both of these quantities will change the rate of increase of the mass of component 1 locally (as defined by the cross-section operator $\langle\ \rangle$).

Terms such as $\langle \bar{\rho}_1 \overline{\alpha' V_{1z}'} \rangle$ could be considered as mass flux terms. Fluctuations in ρ_1, α, V_{1z} and their correlations will change the local mass flux.

It can be seen that it is not easy to determine what the physical significance of each term is in the effective generation function. To gain insight into the physical significance of the fluctuating terms, it is more convenient to look at the terms in their general control volume format.

6. SUMMARY

The cross section averaged form of the mass, energy, and momentum conservation equations in terms of time-averaged variables are given below for a differential length of channel of constant cross-sectional area.

6.1 Continuity Equation

Vapour Phase

$$\frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} > + \frac{\partial}{\partial z} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} > = < \bar{\Gamma}_{1m} >$$

Liquid Phase

$$\frac{\partial}{\partial t} < \bar{\rho}_2 (1-\bar{\alpha}) > + \frac{\partial}{\partial z} < \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} > = - < \bar{\Gamma}_{1m} >$$

where; $< \bar{\Gamma}_{1m} >$ = vapour generation function

$< \bar{\Gamma}_{2m} > = - < \bar{\Gamma}_{1m} >$ for a system without mass addition through the walls

6.2 Momentum Equation

To avoid a detailed description of interphase momentum and energy exchange mechanisms, the momentum equation can be written for the two-phase mixture as

$$\frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} > + \frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z}^2 + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z}^2 >$$

$$= - \left\{ \frac{\partial P}{\partial z} + \frac{\tau_\omega P_\omega}{A} + g < \bar{\rho} > \cos\theta \right\}$$

where

τ_ω = wall shear stress

$< \bar{\rho} > = < \bar{\alpha} \bar{\rho}_1 + (1-\bar{\alpha}) \bar{\rho}_2 >$

6.3 Energy Equation

Similarly the energy equation for the mixture is

$$\frac{\partial}{\partial t} < \bar{\rho}_1 \bar{\alpha} \bar{i}_1 + (1-\bar{\alpha}) \bar{\rho}_2 \bar{i}_2 > + \frac{\partial}{\partial z} < \bar{\rho}_1 \bar{\alpha} \bar{V}_{1z} \bar{i}_1 + \bar{\rho}_2 (1-\bar{\alpha}) \bar{V}_{2z} \bar{i}_2 > = \frac{\dot{q}'' P_h}{A} + \frac{dw}{A\,dz}$$

when the energy is represented by the enthalpy of the vapour and the liquid.

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