1 APPENDIX 2

STRESS IN A FLUID

1.1 Linear momentum principle

The linear momentum principle states that:

The time rate of change of momentum of a body	=	The sum of the forces acting on this body.

For the present application, the body referred to in the above statement consists of some fixed quantity of fluid contained in a material volume, $V_m(\tau)$, limited by a material surface, $A_m(\tau)$, as shown in Figure 1. The momentum of this material volume is:

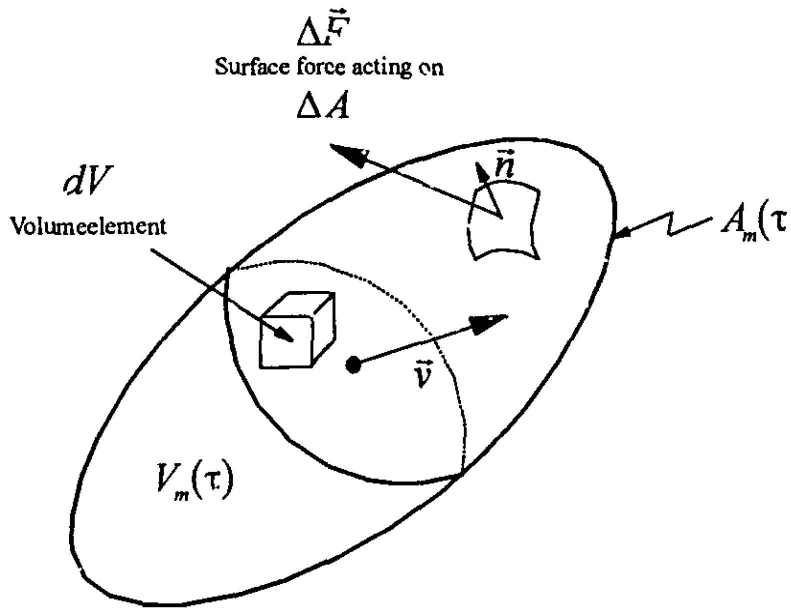

Figure 1 Material volume.

$$\int_{V_m(\tau)} \rho \, \vec{v} \, dV \, . \tag{1}$$

The above linear momentum principle can then be written as:

$$\frac{d}{d\tau} \int_{V_m(\tau)} \rho \, \vec{v} \, dV = \sum \vec{F} \tag{2}$$

where $d/d\tau$ is material derivative. The forces, $\sum \vec{F}$, acting on the material volume consist of body forces (usually gravitational forces) and surface forces. Representing by $\vec{g}$ the body force per unit mass, the total body force is given by:

$$\int_{V_m(\tau)} \rho \vec{g} \, dV. \tag{3}$$

For our application, $\vec{g}$ is the acceleration of the gravity.

The surface forces are handled in terms of the stress vector, $\vec{t}_{(n)}$, acting on a surface having a normal $\vec{n}$ defined as:

$$\vec{t}_{(n)} = \lim_{\Delta A \to 0} \left(\frac{\Delta \vec{F}}{\Delta A} \right).$$

The total surface force acting on the material volume is then written as:

$$\int_{A_m(\tau)} \vec{t}_{(n)} \, dA. \tag{4}$$

Substituting Eqs. 2, 3 and 4 into the linear momentum principle, the following equation results:

$$\frac{d}{d\tau} \int_{V_m(\tau)} \rho \vec{v} \, dV = \int_{V_m(\tau)} \rho \vec{g} \, dV + \int_{A_m(\tau)} \vec{t}_{(n)} \, dA; \tag{5}$$

for a fluid at rest the above equation becomes:

$$0 = \int_{V_m(\tau)} \rho \vec{g} \, dV + \int_{A_m(\tau)} \vec{t}_{(n)} \, dA. \tag{6}$$

1.2 Stress Vector in a Stagnant Fluid

It is well known that a fluid will deform continuously under the effect of shear stresses and it can only be compressed. In the light of these observations, the stress on a surface within a stagnant fluid always acts in the direction opposite to that of the normal to this surface. In the following, considering a fluid element in the shape of a tetrahedron and Eq. 6, we will prove that the stress at a given point in a stagnant fluid is isotropic. Fig. 2 shows the selected fluid element as well as forces acting on its bounding surfaces. Observe that all these forces are normal to the bounding surfaces and they are in the direction opposite to that of the outwardly directed unit normals. Table 1 summarises the forces and outwardly directed unit normals for the four planes limiting the tetrahedron. Application of Eq. 6 to the selected fluid element yields:

$$<\rho \vec{g}>_3 \Delta V - \vec{n} < p_n >_2 \Delta A_n + \vec{i} < p_x >_2 \Delta A_x + \vec{j} < p_y >_2 \Delta A_y + \vec{k} < p_z >_2 \Delta A_z = 0 \tag{7}$$

$< \ >_2$ and $< \ >_3$ show surface and volume averages of a parameter and defined respectively as:

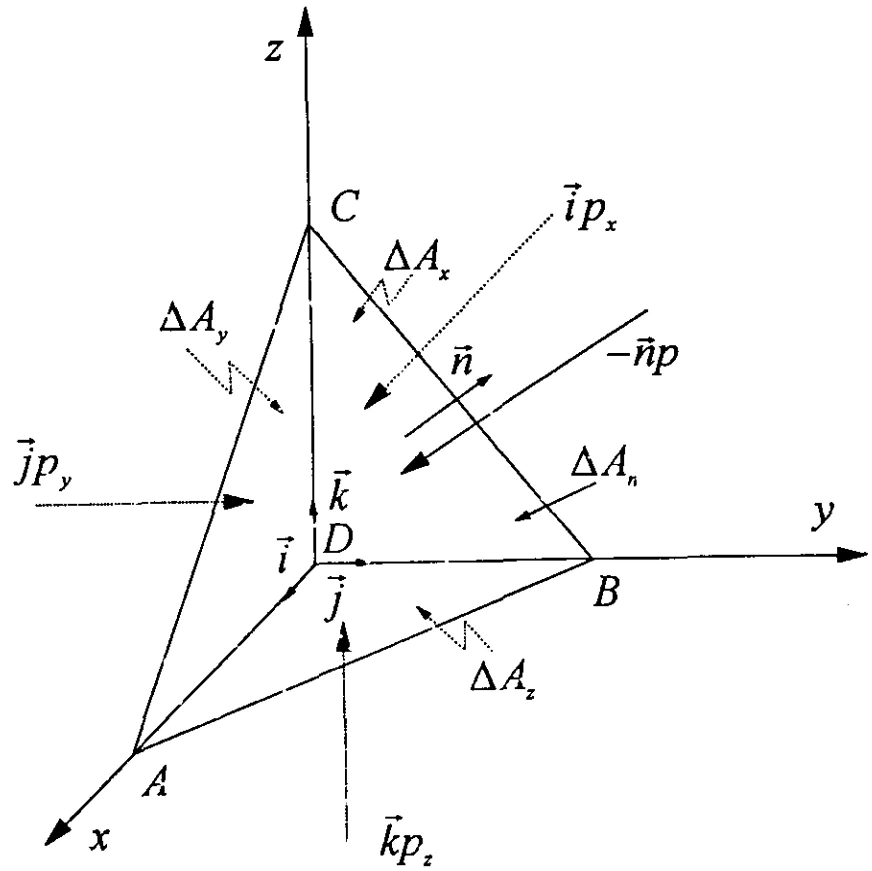

Figure 2 Static stress on a tetrahedron.

$$<f>_2 = \frac{\int_A f dA}{\int_A dA} \tag{8}$$

and

$$<f>_3 = \frac{\int_V f dV}{\int_V dV}. \tag{9}$$

TABLE 1

STATIC STRESS ON A TETRAHEDRON

PLANE	AREA	NORMAL	FORCE VECTOR
ABC	ΔA_n	$\vec{n}$	$-\vec{n}\, p_n$
BCD	ΔA_x	$-\vec{i}$	$\vec{i}\, p_x$
ADC	ΔA_y	$-\vec{j}$	$\vec{j}\, p_y$
ABD	ΔA_z	$-\vec{k}$	$\vec{k}\, p_z$

Referring to the Fig. 2, it can be seen that in Eq. 7 $\Delta A_x, \Delta A_y$ and ΔA_z can be substituted with:

$$\Delta A_x = \vec{i} \cdot \vec{n}\, \Delta A_n = n_x \Delta A_n \tag{10}$$

$$\Delta A_y = \vec{k} \cdot \vec{n}\, \Delta A_n = n_y \Delta A_n \tag{11}$$

$$\Delta A_z = \vec{k} \cdot \vec{n}\, \Delta A_n = n_z \Delta A_n \tag{12}$$

where n_x, n_y and n_z are the cosines directors of the plane ABC; thus Eq. 7 become:

$$< \rho\, \vec{g} >_3 \Delta V - \Delta A_n (\vec{n} < p_n >_2 - \vec{i}\, n_x < p_x >_2 - \vec{j}\, n_y < p_y >_2 - \vec{k}\, n_z < p_z >_2) = 0\,. \tag{13}$$

Dividing the above equation by ΔA_n and taking the limit as $\Delta A_n \to 0$, we will observe that the ratio $\Delta V / \Delta A_n$ tends to zero and Eq. 13 becomes:

$$\lim_{\Delta A_n \to 0} \left[< \rho\, \vec{g} >_3 \frac{\Delta V}{\Delta A_n} - (\vec{n} < p_n >_2 - \vec{i}\, n_x < p_x >_2 - \vec{j}\, n_y < p_y >_2 - \vec{k}\, n_z < p_z >_2 \right]$$

$$= -\vec{n}\, p_n + \vec{i}\, n_x p_x + \vec{j}\, n_y p_y + \vec{k}\, n_z p_z = 0\,. \tag{14}$$

Because of the limiting process ($\Delta A_n \to 0$ or $\Delta V \to 0$), in the above equation the average values have been replaced with point values. Expressing the unit normal in terms of its scalar components:

$$\vec{n} = \vec{i}\, n_x + \vec{j}\, n_y + \vec{k}\, n_z \tag{15}$$

and substituting it into Eq. 14, we obtain:

$$\vec{i}\, n_x (p_n - p_x) + \vec{j}\, n_y (p_n - p_y) + \vec{k}\, n_z (p_n - p_z) = 0\,. \tag{16}$$

In order to satisfy the above equation, all scalar components should be zero, i.e.:

$$p_n = p_x = p_y = p_z\,. \tag{17}$$

The above result proves that the stress at a given point in a stagnant fluid is isotropic. Dropping the subscript *n*, the stress vector acting on any arbitrary surface with outwardly directed normal $\vec{n}$ can be expressed as:

$$\vec{t}_{(n)} = -\vec{n}\, p\,. \tag{18}$$

This equation indicates that the magnitude of the stress vector is equal to the pressure and it is oriented in the opposite direction to that of the unit normal.

1.3 Stress in a Moving Fluid - Stress Vector and Stress Tensor

When fluids move, the study of the surface forces are more difficult. The aim of this section is to discuss the basic nature of the stress vector, $\vec{t}_{(n)}$. During this review we will assume that ρ, $\vec{v}$ and $\vec{t}_{(n)}$ are continuous function of the space and time. Furthermore, we will assume that the stress vector is a continuous function of the orientation of the surface element which is identified by the outwardly oriented unit normal. The following properties of the stress vector and stress tensor will be proved:

1. The stress vectors which act on both sides of a surface at a given point are equal in magnitude and opposite in direction.
2. The stress vector may be expressed in terms of a stress tensor $\overline{\overline{T}}$ as:

$$\vec{t}_n = \vec{n} \cdot \overline{\overline{T}} \ .$$

3. The stress tensor is symmetric:

$$T_{ij} = T_{ji} \ .$$

In order to prove the above statements, we will use appropriate material volumes as well as average values defined with Eqs. 8 and 9.

1.3.1 The relationship between the stress vectors acting on both sides of a surface at a given point

Let us consider a material volume in the shape of a thin disk as illustrated in Fig 3 and indicate by $A_n(\tau)$ the area of each parallel surface, L the thickness of the disk and by $A_L(\tau)$ the area of the lateral surface. $\vec{n}$ and $\vec{l}$ show unit normals to the parallel and lateral surfaces respectively. Applying the linear momentum conservation principle (Eq. 5) to the above material volume we obtain:

$$\frac{d}{d\tau}\int_{V_m(\tau)} \rho\,\vec{v}\,dV = \int_{V_m(\tau)} \rho\,\vec{g}\,dV + \int_{A_L(\tau)} \vec{t}_{(l)}\,dA + \int_{A_n(\tau)} \vec{t}_{(n)}\,dA + \int_{A_n(\tau)} \vec{t}_{(-n)}\,dA \quad (19)$$

or using the volume and surface averages defined by Eqs. 8 and 9, we obtain:

$$\frac{d}{d\tau}\left[<\rho\,\vec{v}>_3 LA_n(\tau)\right] = <\rho\,\vec{g}>_3 LA_n(\tau) + <\vec{t}_{(l)}>_2 A_l(\tau) + \int_{A_n(\tau)}\left[\vec{t}_{(n)} + \vec{t}_{(-n)}\right]dA \ . \quad (20)$$

Taking the limit of the above equation as L goes to zero and observing that $A_l(\tau)$ also goes to zero, we get:

$$\lim_{L\to 0}\int_{A_n(\tau)}\left[\vec{t}_{(n)} + \vec{t}_{(-n)}\right]dA = 0 \ . \quad (21)$$

Since this equation is true for any arbitrary surface element, we conclude that the integrand should be equal to zero. Therefore we obtain:

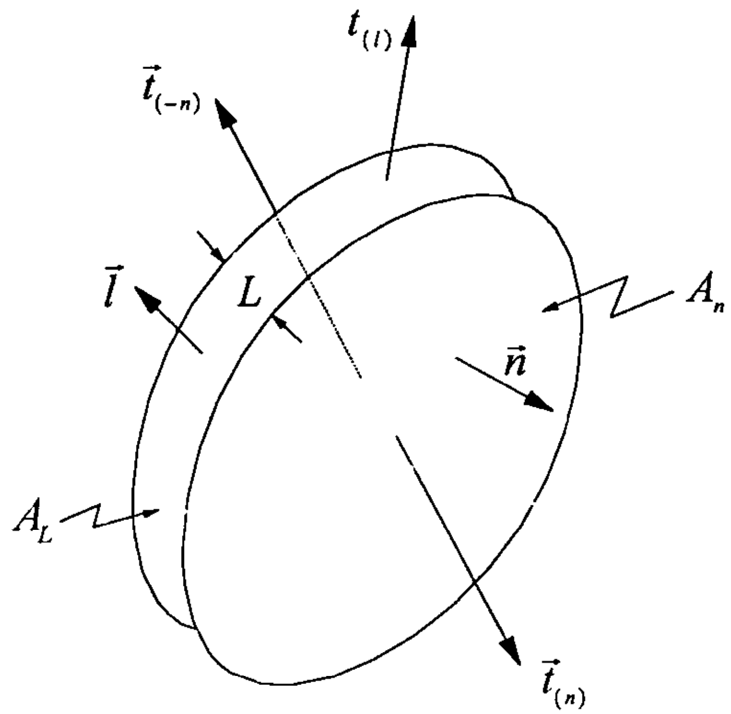

Figure 3 Material volume having the form of an arbitrary slab.

$$\vec{t}_{(n)} = -\vec{t}_{(-n)} \ . \tag{22}$$

This the proof of the 1st statement.

1.3.2 The stress tensor

Let us consider an arbitrary material volume which can be expressed as:

$$V_m(t) = \Gamma(\tau)L^3 \tag{23}$$

where L is a characteristic dimension and $\Gamma(\tau)$ is the shape factor. Using the linear momentum principle (Eq. 5) and volume average defined by Eq. 9 we obtain:

$$\frac{d}{d\tau}\left[<\rho\, \vec{v}>_3 \Gamma(\tau)L^3\right] =<\rho\, \vec{g}>_3 \Gamma(\tau)L^3 + \int_{A_m(\tau)} \vec{t}_{(n)}\, dA \ . \tag{24}$$

Dividing the above equation by L^2 and taking its limit as L goes to zero, we obtain:

$$\lim_{L\to 0}\frac{1}{L^2}\int_{A_m(\tau)} \vec{t}_{(n)}\, dA = 0 \tag{25}$$

which shows that at every point in the space, the stress is in equilibrium. In the following, using the above equation we will establish a relationship between the stress vector and the stress tensor. To this aim, let us consider a volume element in the shape of a tetrahedron as shown in Fig. 4. The stresses acting on the four faces of this volume element as well as the areas associated with them are shown in Table 2. The application of Eq. 25 to the selected volume element with the assumption that L^2 is equal to the oblique area, ΔA_n , yields:

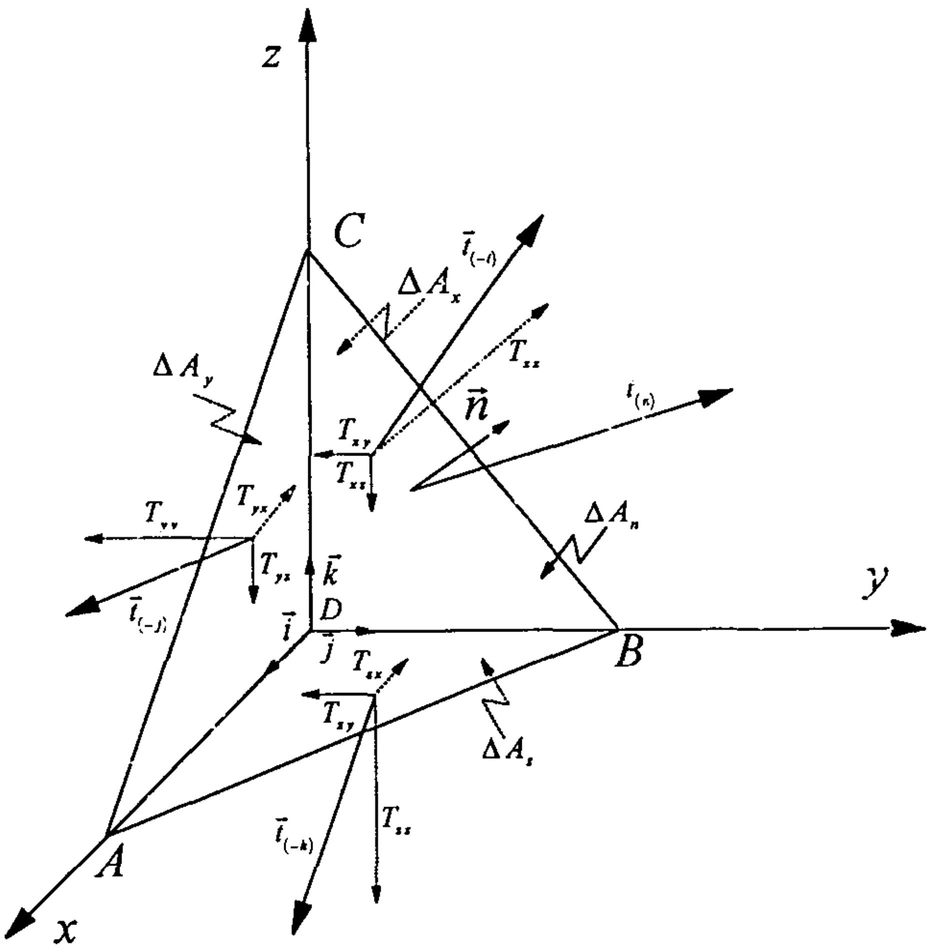

Figure 4 A material volume with surface stresses.

$$\lim_{\Delta A_n \to 0}\left[\frac{1}{\Delta A_n}\left(\Delta A_n < \vec{t}_{(n)} >_2 + \vec{n}\,.\,\vec{i}\;\Delta A_n < \vec{t}_{(-i)} >_2 + \vec{n}\,.\,\vec{j}\;\Delta A_n < \vec{t}_{(-j)} >_2 + \vec{n}\,.\,\vec{k}\;\Delta A_n < \vec{t}_{(-k)} >_2\right)\right] = 0 \qquad (26)$$

TABLE 2

STRESSES ACTING ON THE ABOVE MATERIAL VOLUME (TETRAHEDRON)

PLANE	AREA	NORMAL	STRESS VECTOR
ABC	ΔA_n	$\vec{n}$	$\vec{t}_{(n)}$
BCD	$\vec{n}\,.\,\vec{j}\,\Delta A_n$	$-\vec{i}$	$\vec{t}_{(-i)}$
ADC	$\vec{n}\,.\,\vec{j}\,\Delta A_n$	$-\vec{j}$	$\vec{t}_{(-j)}$
ABD	$\vec{n}\,.\,\vec{j}\,\Delta A_n$	$-\vec{k}$	$\vec{t}_{(-k)}$

If we carry out in Eq. 26 the division by ΔA_n, we obtain:

$$\lim_{\Delta A_n \to 0}\left[<\vec{t}_{(r)}>_2 + \vec{n}\,.\,\vec{i}<\vec{t}_{(-i)}>_2 + \vec{n}\,.\,\vec{j}<\vec{t}_{(-j)}>_2 + \vec{n}\,.\,\vec{k}<\vec{t}_{(-k)}>_2\right] = 0\;. \qquad (27)$$

In the above equation, taking the limit as ΔA_n goes to zero is equivalent to taking the limit as the characteristic length L goes to zero and during this limit taking process all average values tend to local values. Thus, Eq. 27 becomes:

$$\vec{t}_n = -\left[(\vec{n}\,.\,\vec{i})\,\vec{t}_{(-i)} + (\vec{n}\,.\vec{j})\vec{t}_{(-j)} + (\vec{n}\,.\,\vec{k})\,\vec{t}_{(-k)}\right] \quad . \qquad (28)$$

Using Eq. 22 the above equation takes the following form:

$$\vec{t}_{(n)} = \vec{n}\,.\left[\vec{i}\,\vec{t}_{(i)} + \vec{j}\,\vec{t}_{(j)} + \vec{k}\,\vec{t}_{(k)}\right] \qquad (29)$$

where the unit normal $\vec{n}$ is defined by Eq. 15.

Eq. 28 show that the stress vector acting on a surface with unit normal $\vec{n}$ can be expressed in terms of the stress vectors acting on the three co-ordinate planes: $\vec{t}_{(i)}$, $\vec{t}_{(j)}$ *and* $\vec{t}_{(k)}$. In rectangular co-ordinates these stress vectors are expressed as:

$$\vec{t}_{(i)} = \vec{i}\; T_{xx} + \vec{j}\; T_{xy} + \vec{k}\; T_{xz} \qquad (30)$$

$$\vec{t}_{(j)} = \vec{i}\; T_{yx} + \vec{j}\; T_{yy} + \vec{k}\; T_{yz} \qquad (31)$$

$$\vec{t}_{(k)} = \vec{i}\; T_{zx} + \vec{j}\; T_{zy} + \vec{k}\; T_{zz} \qquad (32)$$

where the first subscript indicates the plane upon which the stress acts and the second subscript indicates the direction in which the stress acts. Taking into account that in Eq. 29 the products:

$$\vec{i}\,\vec{t}_{(i)}\ ,\ \vec{j}\,\vec{t}_{(j)}\ and\ \vec{k}\,\vec{t}_{(k)}$$

are diadic product of two vectors and combining Eqs. 29, 30, 31, 32 and 15 we obtain:

$$\vec{t}_{(n)}=\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}\times\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} T_{xx} \\ T_{xy} \\ T_{xz} \end{bmatrix}+\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}\times\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} T_{yx} \\ T_{yy} \\ T_{yz} \end{bmatrix}+\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}\times\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} T_{zx} \\ T_{zy} \\ T_{zz} \end{bmatrix}$$

or

$$\vec{t}_{(n)}=\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}\times\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}=\vec{n}\,.\,\bar{\bar{T}} \qquad (33)$$

where $\bar{\bar{T}}$ is the stress tensor. Therefore, we just proved that the stress vector acting on any arbitrary surface can be obtained by multiplying the stress tensor with the unit normal to this surface.

1.4 The symmetry of the Stress Tensor

Using a material volume which has the shape of a differential cube as shown in Fig. 5 and the angular momentum equation which has the following form:

$$\frac{d}{d\tau}\int_{V_m(\tau)}\left(\vec{r}\times\rho\,\vec{v}\right)dV=\int_{V_m(\tau)}\left(\vec{r}\times\rho\,\vec{g}\right)dV+\int_{A_m(\tau)}\left(\vec{r}\times\vec{t}_{(n)}\right)dA \qquad (34)$$

we will prove the symmetry of the stress tensor. In the above equation $\vec{r}$ is the lever arm and $V_m(\tau)$ is an arbitrary material volume surrounded a material surface $A_m(\tau)$. Assuming that this arbitrary material volume can be again represented by:

$$V_m(\tau)=\Gamma(\tau)L^3 \qquad (35)$$

and using the volume average defined by Eq. 9, Eq. 34 becomes:

$$\frac{d}{d\tau}\left[<\vec{r}\times\rho\,\vec{v}>_3\Gamma(\tau)L^3\right]=<\vec{r}\times\rho\,\vec{g}>_3\Gamma(\tau)L^3+\int_{A_m(t)}\left(\vec{r}\times\vec{t}_{(n)}\right)dA\ . \qquad (36)$$

Let us return now to the cubical volume element illustrated in Fig. 5 and assume that the characteristic length L is:

$$L=\Delta x=\Delta y=\Delta z.$$

and that the origin of the co-ordinates O coincide with on of the corner of this element. Thus for the selected volume element, the position vector $\vec{r}$ appearing in Eq. 36 defines any point in this element and it tends to zero as L tends to zero. Dividing Eq. 36 by L^3 and taking its limit as L

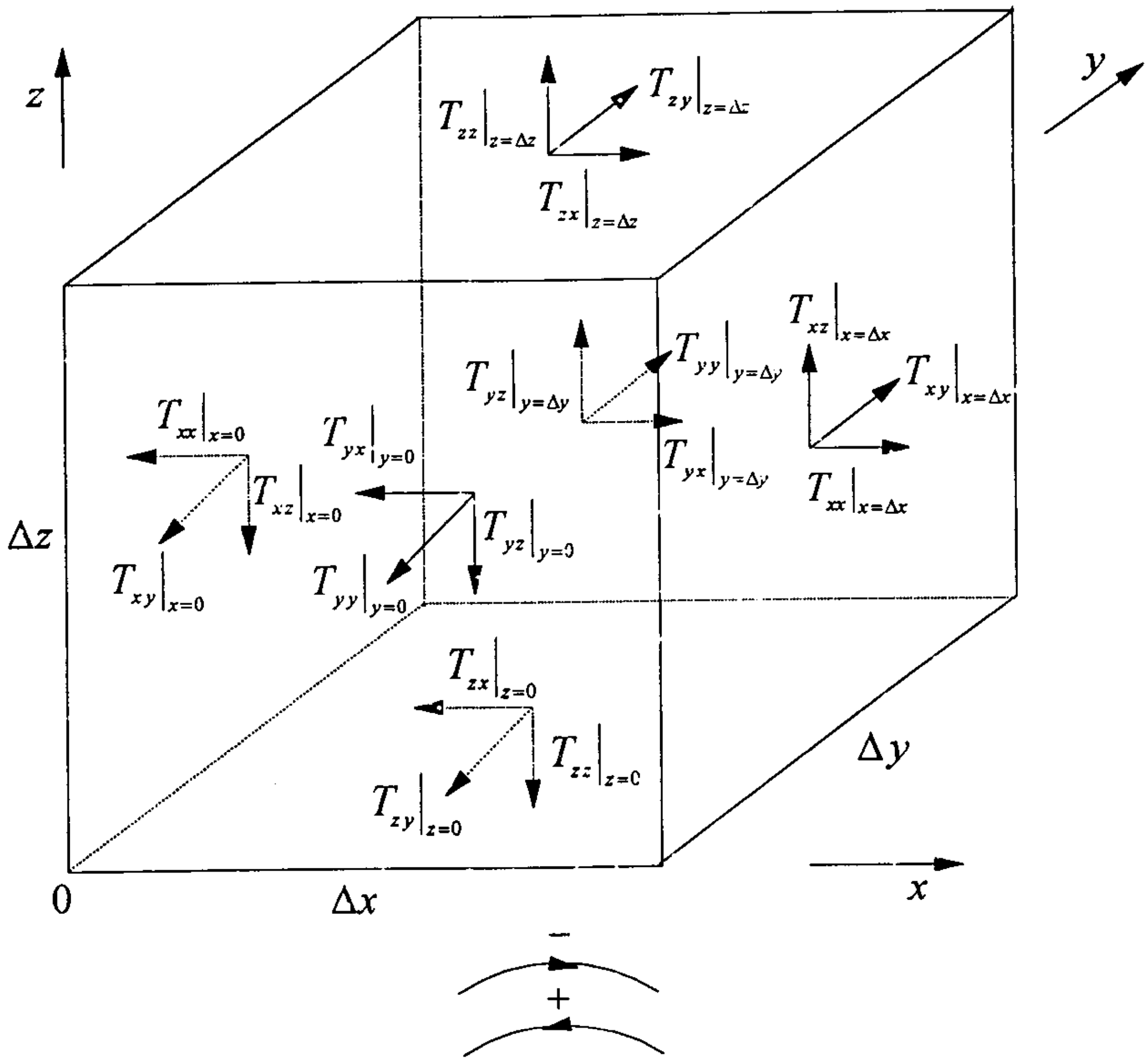

Figure 5 Stress on a differential cube - symmetry of a stress tensor.

goes to zero we obtain:

$$\lim_{L\to 0}\frac{1}{L^3}\int_{A_m(t)}\left(\vec{r}\times t_{(n)}\right)dA = 0 \tag{37}$$

The above equation shows that the torques are in local equilibrium. The multiplication of Eq. 37 by $\vec{i}, \vec{j}$ and $\vec{k}$ gives the X, Y and Z components of the torque. For example, Z component of the torque for the selected volume element is as follows:

$$\lim_{L\to 0}\frac{1}{L^3}\left\{\underbrace{\frac{L}{2}(T_{xx_{x=0}}L^2)-\frac{L}{2}(T_{xx_{x=\Delta x}}L^2)+L(T_{xy_{x=\Delta x}}L^2)}_{X\text{-Surfaces}}\right.$$

$$\underbrace{-\frac{L}{2}(T_{yy_{y=0}}L^2)+\frac{L}{2}(T_{yy_{y=\Delta y}}L^2)-L(T_{yx_{y=\Delta y}}L^2)}_{Y\text{-Surfaces}}$$

$$\left.\underbrace{+\frac{L}{2}(T_{zx_{x=0}}L^2)-\frac{L}{2}(T_{zx_{x=\Delta x}}L^2)-\frac{L}{2}(T_{zy_{y=0}}L^2)+\frac{L}{2}(T_{zy_{y=\Delta y}})L^2}_{Y\text{-Surfaces}}\right\}=0\ . \tag{38}$$

In the above equation, the averaging operator $< \ >_2$ has been omitted to simplify the presentation. We can easily observe that when L tends to zero:

$$T_{xx_{x=0}} = T_{xx_{x=\Delta x}}\ ;\ T_{yy_{y=0}} = T_{yy_{y=\Delta y}}\ ;\ T_{zx_{y=0}} = T_{zx_{x=\Delta x}}\ \text{and}\ T_{zy_{y=0}} = T_{zy_{y=\Delta y}}$$

therefore we obtain:

$$\lim_{L\to 0}\{T_{xy_{x=\Delta x}} - T_{yx_{y=\Delta y}}\} = 0 \tag{39}$$

or

$$T_{xy} = T_{yx}\ . \tag{40}$$

If we consider X and Y components of Eq. 37 we will find that:

$$T_{yz} = T_{zy} \tag{41}$$

and

$$T_{xz} = T_{zx}\ . \tag{42}$$

1.5 Viscous Stress Tensor

We have seen that stagnant fluids experience only normal stress and at a given point this stress is isotropic and oriented in the direction opposite to that of the unit normal of the surface on which it acts. We called the magnitude of this stress "the pressure" and it is given with Eq. 18. Using the definition of the unit tensor given with Eq. 3 in Appendix 1, Eq. 18 can also be written as:

$$\vec{t}_{(n)} = -\vec{n}\,p = -\vec{n}\,.\,\overline{\overline{I}}\,p\ . \tag{44}$$

We can easily see that the left and right product of the unit tensor with a vector are equal and they are also equal to the vector itself:

$$\vec{V} \,.\, \bar{\bar{I}} = \bar{\bar{I}} \,.\, \vec{V} = \vec{V} \tag{45}$$

In moving fluids, besides the viscous forces due to the motion of the fluid (deformation of the fluid) the pressures forces are also present. Therefore, it is convenient to split stress tensor into two parts: one representing the pressure stress and the other viscous stress:

$$\bar{\bar{T}} = -p\,\bar{\bar{I}} + \bar{\bar{\sigma}} \tag{46}$$

and the stress vector will be written as:

$$\vec{t}_{(n)} = \vec{n} \,.\left(-p\,\bar{\bar{I}} + \bar{\bar{\sigma}}\right) \ . \tag{47}$$

Since $\bar{\bar{T}}$ and $\bar{\bar{I}}$ are symmetric tensors, $\bar{\bar{\sigma}}$ also should be a symmetric tensor. Furthermore, since all of the off-diagonal terms of $p\,\bar{\bar{I}}$ are zero, the all off-diagonal terms of $\bar{\bar{T}}$ and $\bar{\bar{\sigma}}$ should be equal:

$$T_{xy} = \sigma_{xy} \ ; \ T_{yz} = \sigma_{yz} \text{ and } T_{zx} = \sigma_{zx} \ .$$