1 APPENDIX 1

TENSORS

We will use the tensorial calculus only to simplify the presentation of the equations. None of their properties will be used. This justifies, in this review, the absence the mathematical definition of the tensors which we will consider as a square array of nine elements:

$$\bar{\bar{T}} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = (t_{ij}) \tag{1}$$

i and j show respectively rows and columns.

The sum of two tensors:

$$\bar{\bar{T}} + \bar{\bar{J}} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{22} & s_{33} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = t_{ij} + s_{ij}$$

$$= \begin{pmatrix} t_{11}+s_{11} & t_{21}+s_{21} & t_{31}+s_{31} \\ t_{21}+s_{21} & t_{22}+s_{22} & t_{23}+s_{23} \\ t_{33}+s_{33} & t_{32}+s_{32} & t_{33}+s_{33} \end{pmatrix}. \tag{2}$$

Unit tensor:

$$\bar{\bar{I}} = (\delta_{ij}) \quad \textit{with} \quad \begin{matrix} \delta_{ij} = 0 & \text{if} & i \neq j \\ \delta_{ij} = 1 & \text{if} & i = j \end{matrix}$$

therefore:

$$\bar{\bar{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Tensorial product of two vectors:

$\vec{V} = (v_i)$ $\quad\vec{W} = (w_j)\quad$ with: $\quad i = 1,2,3 \quad j = 1,2,3$

therefore:

$$\overset{=}{W} = \vec{V}\ \vec{W} = (v_i w_j)$$

or

$$\overset{=}{T} = \begin{pmatrix} v_1w_1 & v_1w_2 & v_1w_3 \\ v_2w_1 & v_2w_2 & v_2w_3 \\ v_3w_1 & v_3w_2 & v_3w_3 \end{pmatrix}. \tag{4}$$

This product is also called the diadic product of two vectors.

Vectorial product of a tensor and a vector

a) Right product:

$$\overset{=}{T}\ .\ \vec{V} = (t_{ik}.v_k) = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} . \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} t_{11}v_1 + t_{12}v_2 + t_{13}v_3 \\ t_{21}v_1 + t_{22}v_2 + t_{23}v_3 \\ t_{31}v_1 + t_{32}v_2 + t_{33}v_3 \end{pmatrix} \tag{5}$$

b) Left product

$$\vec{V}\ .\ \overset{=}{T} = (v_k.t_{ki}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} . \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \begin{pmatrix} v_1t_{11} + v_2t_{21} + v_3t_{31} \\ v_1t_{12} + v_2t_{22} + v_3t_{32} \\ v_1t_{13} + v_2t_{23} + v_3t_{33} \end{pmatrix} \tag{6}$$

The product of a vector and a tensor is a vector. In the above summary, we used the convention of mute index which means that when an index is repeated twice in a monome a summation should be carried out it. If a tensor is symmetrical with respect to its principal diagonal, the left and right products of this tensor with the same vector are equal.

Divergence vector:

The divergence vector is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x_i} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \tag{7}$$

Vectorial divergence of a tensor:

The vectorial divergence of a vector id given by:

$$\vec{\nabla}\ .\ \overset{=}{T} = (\nabla_k.t_{ki})$$

or

$$\vec{\nabla} \,.\, \bar{\bar{T}} = \begin{pmatrix} \dfrac{\partial t_{11}}{\partial x} + \dfrac{\partial t_{12}}{\partial y} + \dfrac{\partial t_{13}}{\partial z} \\ \dfrac{\partial t_{12}}{\partial x} + \dfrac{\partial t_{22}}{\partial y} + \dfrac{\partial t_{32}}{\partial z} \\ \dfrac{\partial t_{13}}{\partial x} + \dfrac{\partial t_{23}}{\partial y} + \dfrac{\partial t_{33}}{\partial z} \end{pmatrix} \tag{8}$$

Tensorial gradient of a vector:

Tensorial gradient of a vector is given by:

$$\vec{\nabla}\vec{V} = (\nabla_i . V_j) = \begin{pmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_3}{\partial x} \\ \dfrac{\partial v_1}{\partial y} & \dfrac{\partial v_2}{\partial y} & \dfrac{\partial v_3}{\partial y} \\ \dfrac{\partial v_1}{\partial z} & \dfrac{\partial v_2}{\partial z} & \dfrac{\partial v_3}{\partial z} \end{pmatrix} \tag{9}$$

Scalar product of two tensors:

The scalar product of two tensor is given by:

$$\bar{\bar{T}} : \bar{\bar{S}} = t_{11}s_{11} + t_{12}s_{12} + t_{13}s_{13} + t_{21}s_{21} + t_{22}s_{22} + t_{23}s_{23} + t_{31}s_{31} + t_{32}s_{32} + t_{33}s_{33} \tag{10}$$