# 4. Forced Convection Heat Transfer

## 4. Forced Convection Heat Transfer

In Chapter 3, we have discussed the problems of heat conduction and used the convection as one of the boundary conditions that can be applied to the surface of a conducting solid. We also assumed that the heat transfer rate from the solid surface was given by Newton's law of cooling:

$$q_{cv} = h_c A\left(t_w - t_f\right) \qquad 4.1$$

In the above application, $h_c$, the convection heat transfer coefficient has been supposed known. The aim of this chapter is to discuss the basis of heat convection in fluids and to present methods (correlations) to predict the value of the convection heat transfer coefficient (or film coefficient).

As already pointed out, the convection is the term used to indicate heat transfer which takes place in a fluid because of a combination of conduction due to molecular interactions and energy transport due to the motion of the fluid bulk. The motion of the fluid bulk brings the hot regions of the fluid into contact with the cold regions. If the motion of the fluid is sustained by a force in the form of pressure difference created by an external device, pump or fan, the term of "forced convection is used". If the motion of the fluid is sustained by the presence of a thermally induced density gradient, then the term of "natural convection" is used.

In both cases, forced or natural convection, an analytical determination of the convection heat transfer coefficient, $h_c$, requires the knowledge of temperature distribution in the fluid flowing on the heated surface. Usually, the fluid in the close vicinity of the solid wall is practically motionless. Therefore, the heat flux from the solid wall can be evaluated in terms of the fluid temperature gradient at the surface:

$$q''_{cv} = -k_f\left(\frac{dt}{dn}\right)_S \qquad 4.2$$

where

$k_f$ : thermal conductivity of the fluid

$\left(\frac{dt}{dn}\right)_S$ : fluid temperature gradient at the surface in the direction of the normal to the surface

The variation of the temperature in the fluid is schematically illustrated in Figure 4.1. Combining Equations 4.1 and 4.2 we obtain:

$$h_c = \frac{-k_f (dt / dn)_S}{(t_w - t_f)} \quad 4.2a$$

where

$t_w$ : temperature of the wall

$t_f$ : temperature of the fluid far from the wall

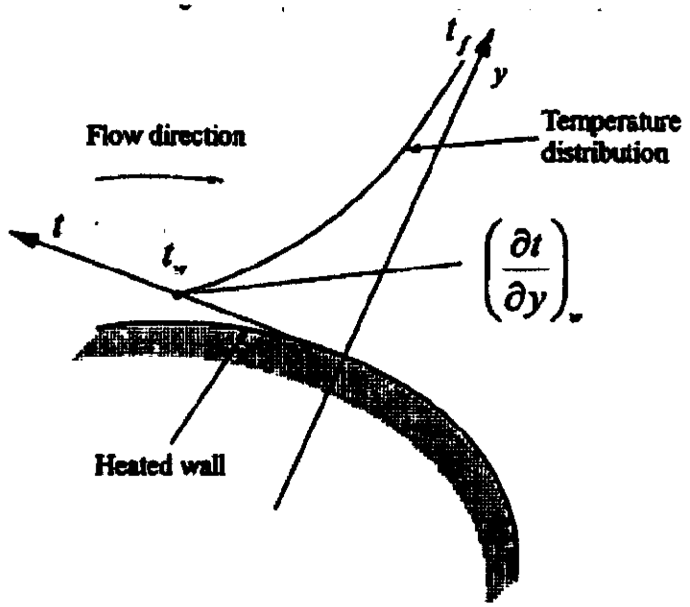



Figure 4.1 Variation of the temperature in the fluid next to the heated surface

The analytical determination of $h_c$ given with Equation 4.2a is quite complex and requires the solution of the fundamental equations governing the motion of viscous fluid; equations of conservations of mass, momentum and energy. A brief discussion of these equations was given in Chapter 2.

## 4.1 Fundamental Aspects of Viscous Motion and Boundary Layer Motion

### 4.1.1 Viscosity

The nature of viscosity is best visualized with the following experiment. Consider a liquid placed in the space between two plates, one of which is at rest, the other moves with a constant velocity $U$ under the effect of a force $F$. The experimental setup is illustrated in Figure 4.2.

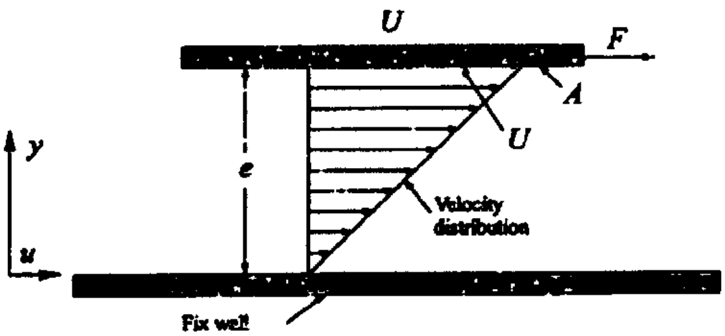



Figure 4.2 Shear stress applied to a fluid

The distance between the plates is $e$ and the surface area of the upper plate in contact with liquid is $A$. Because of the non-slip condition, the fluid velocity at the lower plate is zero and at the upper plate is $U$. Assuming that the Couette flow conditions prevail (ie, no pressure gradient in the flow direction) a linear velocity distribution, as shown in Figure 4.1, develops between the plates and is given by:

$$u = \frac{U}{e} y \qquad 4.3$$

The slope of this distribution is constant and given by:

$$\frac{du}{dy} = \frac{U}{e} \qquad 4.4$$

The shear stress exerted by the plate to the liquid is written as:

$$\tau = \frac{F}{A}$$

It is possible to repeat the above experiment for different forces (i.e. upper plate velocities) and plot the resulting shear stress, $\tau$, versus the slope of the velocity distribution $(du/dy)$. Such a plot is shown in Figure 4.3.

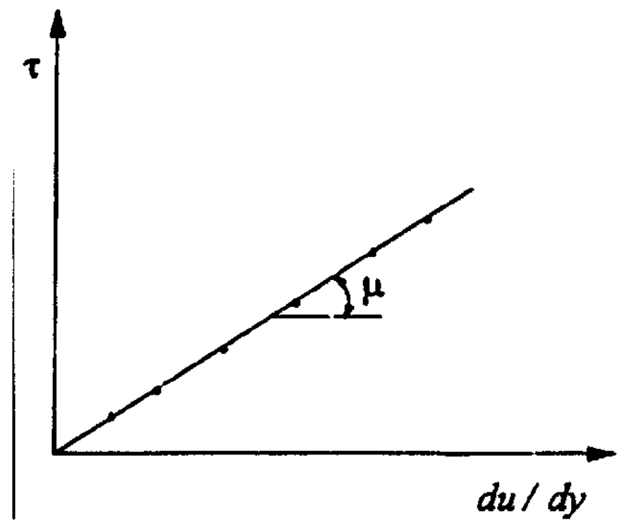



Figure 4.3 $\tau$ versus $(du/dy)$

Data points lie on a straight line that passes through the origin. Therefore $\tau$ is proportional to the velocity gradient, $(du/dy)$ and the constant of proportionality is $\mu$. $\mu$ is called the "dynamic viscosity". Based on the above discussion, the shear stress can be written as:

$$\tau = \mu \frac{du}{dy} \qquad 4.5$$

In a more general way, consider a laminar flow over a plane wall. The velocity of the fluid is parallel to the wall and varies from zero to some value far from the wall. The velocity distribution close to the wall, as depicted in Figure 4.4 is not linear.

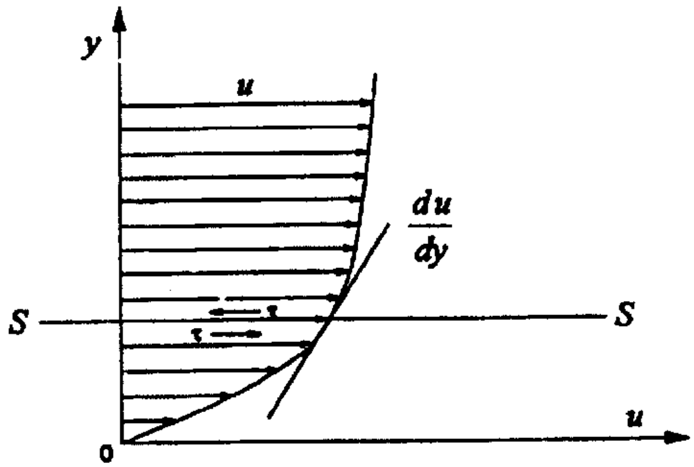



Figure 4.4 Velocity distribution next to a wall

Let us select a plane $SS$ parallel to the wall. The fluid layers on either side of $SS$ experience a shearing force $\tau$ due to their relative motion. The shearing stress, $\tau$, produced by this relative motion is again directly proportional to the velocity gradient in a direction normal to the plane $SS$:

$$\tau = \mu\left(\frac{du}{dy}\right)_{SS} \qquad 4.6$$

The ratio of the dynamic viscosity to the specific mass of the fluid

$$\nu = \frac{\mu}{\rho} \qquad 4.7$$

is called "kinematic viscosity"

The dynamic viscosity has dimensions:

$$\mu = \frac{\tau}{\frac{du}{dy}} = \frac{F}{L^2}\frac{T}{L}L = \frac{FT}{L^2} \qquad 4.8$$

$F,L,T$ are force, length and time, respectively. In the $SI$, the dimensions of the dynamic viscosity becomes:

$$\mu = \frac{N_s}{m^2}$$

The dimensions of the kinematic viscosity are:

$$\nu = \frac{L^2}{T}$$

or in $SI$ units

$$\nu = \frac{m^2}{s}$$

The physical basis of viscosity is the momentum exchange between the fluid layers. To understand better this statement, consider one dimensional laminar flow of a dilute gas on a plane wall as depicted in Figure 4.5. The velocity of the fluid $u$ is only a function of $y$. Let us imagine in the flow a surface $SS$ parallel to the plane wall.

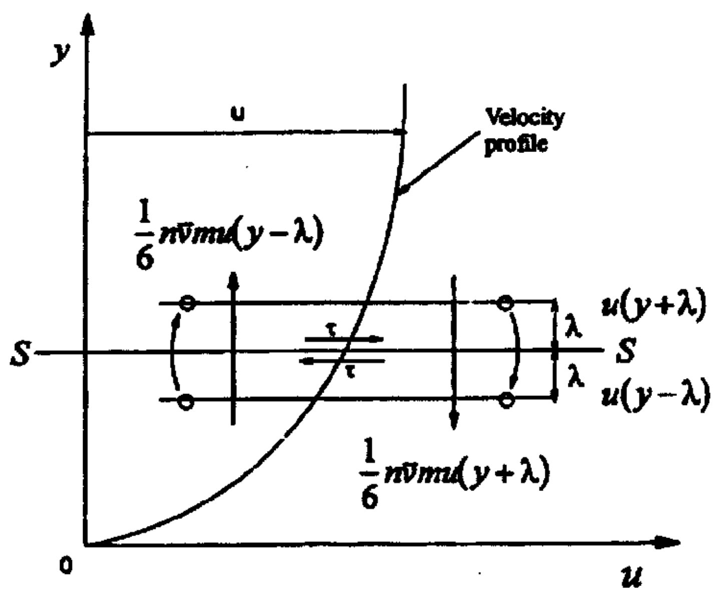



Figure 4.5 Momentum exchange by molecular diffusion

Because of the random thermal velocities, gas molecules continually cross the $SS$ surface both above and below. We may assume that the last collision before crossing the surface $SS$, each molecule acquires the flow velocity corresponding to the height at which this collision has taken place.

Since this velocity above the $SS$ is greater than that below, molecules crossing from above transport a greater momentum in the direction of the flow across the surface than that transported by the molecules crossing the same surface from below. The result is a net transport of momentum across the surface $SS$ from the region above to the region below. According to the Newton's second law, this change of momentum is balanced by the viscous force. This is the reason for which the region of gas above $SS$ is submitted to a force which is due to the region of the gas below $SS$ ($-\tau$) and vice versa ($\tau$).

We will try now to estimate in an approximate manner the dynamic of viscosity $\mu$. If there are $m$ molecules per unit volume of the dilute gas, approximately 1/3 of these molecules have an average velocity ($\bar{v}$) parallel to the y-axis. From these molecules, half of them (ie, $\frac{1}{6}n$) have an average velocity in the direction of $y^+$ and the other half have an average velocity in the direction of $y^-$. Consequently, an average $\frac{1}{6}n\bar{v}$ molecules cross the plane $SS$ per unit surface and per unit time from above to below and vice versa. Molecules coming from above $SS$ undergo their last collision at a distance approximately equal to the mean free path $\lambda$ and their flow velocity is $u(y+\lambda)$ and their momentum is $mu(y+\lambda)$ where $m$ is the mass of the molecule. The same argument is also true for molecules coming from below the surface $SS$ and their velocity is $u(y-\lambda)$ and momentum $mu(y-\lambda)$. Therefore, the momentum component in the direction of the flow that crosses the surface $SS$ from above to below is:

$$\frac{1}{6}n\bar{v}\,[mu(y+\lambda)] \qquad 4.9$$

and from below to above:

$$-\int_A \vec{n}\cdot\rho\vec{v}\vec{v}\,dA - \int_A \vec{n}\cdot p\bar{\bar{I}}\,dA + \int_A \vec{n}\bar{\bar{\sigma}}\,dA = 0\ \frac{1}{6}n\bar{v}\,[mu(y-\lambda)] \qquad 4.10$$

The net momentum transport is the difference between Eqs 4.10 and 4.9 and according to the Newton's second law should be balanced by a viscous force, $\tau$. Therefore we may write:

$$\tau = \frac{1}{6} n\bar{v}m[u(y-\lambda)-(y+\lambda)] \qquad 4.11$$

Developing $u(y-\lambda)$ and $u(y+\lambda)$ in Taylor series and neglecting the terms of second and higher orders, we obtain:

$$u(y+\lambda) \cong u(y) + \lambda\frac{du}{dy} \qquad 4.12$$

$$u(y-\lambda) \cong u(y) - \lambda\frac{du}{dz} \qquad 4.13$$

Substitution of Eqs 4.12 and 4.13 into 4.11 yields.

$$\tau = -\frac{1}{3} n\bar{v}m\lambda\frac{du}{dy} = -\mu\frac{du}{dy} \qquad 4.14$$

The negative sign shows that the viscous stress acting on the upper face of $SS$ surface is in the direction opposite to the flow direction (or $x^+$). From 4.14 we observe that:

$$\mu = \frac{1}{3} n\bar{v}m\lambda \qquad 4.15$$

Although the constant 1/3 may not be correct, the dependence of $\mu$ on $n, \bar{v}, m$ and $\lambda$ should be rather correct.

### 4.1.2 Fluid Conservation Equations – Laminar Flow

We have already pointed out that the analytical determination of the convection heat transfer coefficient defined with Eq. 4.2 requires the solution of the fluid conservation equations: mass, momentum and energy to obtain the temperature distribution in the fluid washing the heated solid. Once the temperature distribution is determined and if the fluid motion in the region immediately adjacent to the heated wall is laminar, which is usually the case, the convection heat transfer coefficient is then determined by using Eq. 4.2. The derivation of the fluid conservation equations is beyond the objective of this course. We will present within the framework of this course, the basic elements which enter in the derivation of the conservation equations and present these equations for an incompressible flow.

In Chapter 2, we have already established that the fluid conservation equations have the following forms:

I) Local mass conservation equation

$$\frac{\partial}{\partial \tau}\rho + \vec{\nabla}.\rho\vec{v} = 0 \qquad 2.6$$

II) Local momentum conservation equation

$$\frac{\partial}{\partial \tau}\rho\vec{v} + \vec{\nabla}\rho\vec{v}\vec{v} = \vec{\nabla}p\bar{I} + \vec{\nabla}\cdot\bar{\bar{\sigma}} + \rho\vec{g} \qquad 4.7$$

III) Energy conservation equation (total energy in enthalpy form)

$$\frac{\partial}{\partial \tau}\rho\left(h + \frac{1}{2}\vec{v}\cdot\vec{v}\right) - \frac{\partial}{\partial \tau}p + \vec{\nabla}\cdot\rho\left(h + \frac{1}{2}\vec{v}\cdot\vec{v}\right)\vec{v}$$
$$= -\vec{\nabla}\cdot\vec{q}'' + \vec{\nabla}\cdot\left(\bar{\bar{\sigma}}\cdot\vec{v}\right) + \rho\vec{g}\cdot\vec{v} + \acute{Q}_g \qquad 4.8$$

In the above equation:

$$\bar{\bar{I}} \text{ is the unit tensor } = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\bar{\bar{\sigma}} \text{ is the stress tensor } = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}$$

$h$ : enthalpy
$p$ : pressure
$\vec{v}$ : velocity ( $u,v,w$ are components of the velocity vector)
$\rho$ : specific mass
$\vec{g}$ : acceleration of gravity
$\vec{q}''$ : heat flux
$\dot{Q}_g$ : energy generation

Each term of the stress tensor can be related to the velocity gradients as follows (Janna, 1986)

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad 4.9$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad 4.10$$

$$\sigma_{zz} = 2\mu\frac{\partial w}{\partial z} + \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad 4.11$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad 4.12$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad 4.13$$

$$\sigma_{zx} = \sigma_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad 4.14$$

Furthermore we know that

$$\vec{q}'' = -k\vec{\nabla}\cdot t \quad 4.15$$

Assuming that

1. The fluid is incompressible and has constant properties, i.e.,

   $c_p$, $\rho$, $\mu$ and $k$ are constant,

2. The kinetic energy and potential energies are negligible,

3. The pressure doesn't change with time, $\frac{\partial p}{\partial \tau}=0$

4. No energy generation $\dot{Q}_g=0$,

and taking into account Eq. 4.9 through 4.15, the conservation equations (Eq's 4.6, 4.7 and 4.8) become:

I. Mass conservation equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4.16}$$

II. Momentum conservation equation

x-component

$$\rho\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x \tag{4.17}$$

y-component

$$\rho\left(\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y \tag{4.18}$$

z-component

$$\rho\left(\frac{\partial w}{\partial \tau}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z \tag{4.19}$$

III. Energy equation

$$\rho c_p\left(\frac{\partial t}{\partial \tau}+u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}+w\frac{\partial t}{\partial z}\right)=k\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}\right)+\mu\phi \tag{4.20}$$

where

$$\phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2$$

$\mu\phi$ is the viscous dissipation term, it is usually negligible compared to heat transferred. Eqs. 4.17 through 4.19 are known as Navier-Stokes equations. In the problems of heat convection is a three-dimensional incompressible laminar flow, the energy equation (Eq. 4.20) must be solved to obtain the temperature distribution in the fluid. However, this equation contains the three components of the velocity and its solution can only be carried out in conjunction with the mass conservation equations and Navier-Stokes equations for a given set of boundary conditions: shape and temperature of the heated body over which the fluid flows, the fluid velocity and temperature far from the body, etc. For the incompressible and constant property fluid we selected the unknown quantities are: $u$, $v$, $w$, $p$ and $t$. There are five equations: 4.16 through 4.20 to determine these unknowns. Once the temperature distribution in the fluid is known, the convection heat transfer coefficient at a given point on the heated surface can be determined with the aid of Eq. 4.2.

It should be pointed out that the basic equations that govern the convection are non-linear and are among the most complex equations of applied mathematics. No general methods are available for the solution of these equations. Analytical solutions exist for very simple cases. In recent years, with the advent of high-speed and high-capacity computers, a good deal of progress has been made in the analysis of complicated heat transfer problems. However, these analysis is time consuming and very costly. Fortunately, a large number of engineering problems can be adequately handled by using simplified forms of the conservation equations, ie., by using a one dimensional model and experimentally determined constitutive equations such as friction and heat transfer coefficients. The solution of these simplified forms can be obtained more easily. On the other hand, the soundness if the assumptions made to obtain the simplified forms of the conservation equations should be verified by ad-hoc experiments.

The conservation equations derived above apply to a laminar fluid motion. In laminar flows, the fluid particles follow well-defined streamlines. These streamlines remain parallel to each other and they are smooth. Heat and momentum are transferred across the streamlines (or between the adjacent fluid layers which slide relative to one another) only by molecular diffusion as described in Section 4.1. Therefore, the cross flow is so small that when a coloured dye is injected into the fluid at some point, it follows the streamline without an appreciable mixing. Laminar flow exists at relatively low velocities.

## 4.3 Fluid Conservation Equation – Turbulent Flow

The term turbulent is used to indicate that there are random variations or fluctuations of the flow parameters such as velocity, pressure and arid temperatures about a mean value. Figure 4.6 illustrates, for example, the fluctuations of the $u$ component of the velocity obtained from a hot-wire anemometer.

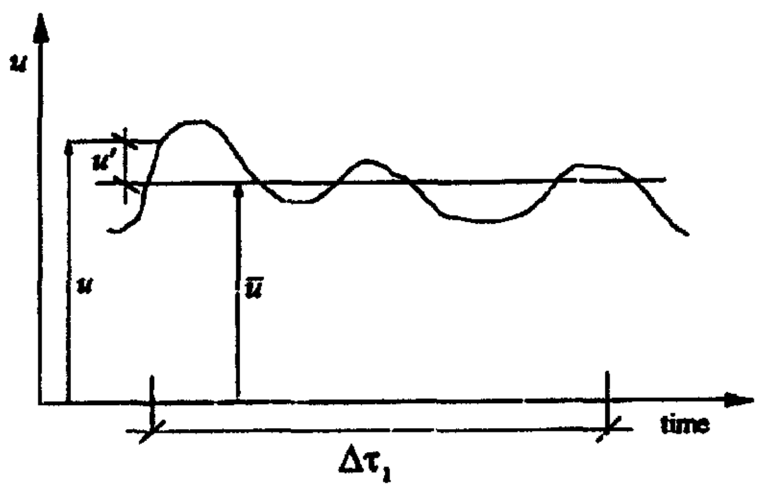



Figure 4.6 Turbulent velocity fluctuations about a time average.

If we denote by $\bar{u}$ the time average velocity and by $u'$ the time dependent velocity fluctuation about that average value, the true velocity may be written as:

$$u = \bar{u} + u' \tag{4.21}$$

The same reasoning may also be done for two components and write:

$$v = \bar{v} + v' \quad 4.22$$

$$w = \bar{w} + w' \quad 4.23$$

Because of these randomly fluctuating velocities, the fluid particles do not stay in one layer but move tortuously throughout the flow. This means that a certain amount of mixing and energy exchange occurs between the fluid layers due to the random motion of fluid particles. This type of mixing is non-existent in laminar flows and is of great importance in heat transfer problems since the random motion of the particles tend to increase the rate of heat exchange between the fluid layers. As a matter of fact, the rate of heat transfer is generally much higher in turbulent flows than in laminar flow. Turbulent flow exists at velocities that are much higher than in laminar flow. As a final note, the pressures and temperatures in turbulent flows are written as:

$$p = \bar{p} + p' \quad 4.24$$

$$t = \bar{t} + t' \quad 4.25$$

The average values appearing in Eqs. 4.21 through 4.25 for steady turbulent motion are given by:

$$\bar{f} = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} f d\tau, \text{ independent of time for steady flows} \quad 4.26$$

The time interval $\Delta\tau$, is taken large enough to exceed amply the period of the fluctuations. On the other hand, the time average of the fluctuations, $f'$is zero:

$$\bar{f'} = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} f' d\tau = \frac{1}{\Delta\tau_1}\int_0^{\Delta\tau_1} (f - \bar{f})d\tau = \bar{f} - \bar{f} = 0 \quad 4.27$$

In the above discussion $f$ may denote any flow parameter.

To obtain the fluid conservation equations which apply to a turbulent flow, we substitute in Eqs. 4.16 through 4.20, $u, v, w, p$ *and* $t$ by Eqs. 4.21, 4.22, 4.23, 4.24 and 4.25 respectively. Since in most of the convection problems the viscous dissipation is negligible, this term can be dropped from the energy conservation equation (Eq. 4.20). The next step consists of time averaging of the resulting equations by taking into account the following averaging rules:

$$f=\bar{f}+f' \qquad 4.28$$

$$g=\bar{g}+g' \qquad 4.29$$

$$\bar{f'}=\bar{g'}=0 \qquad 4.30$$

$$\overline{f+g}=\bar{f}+\bar{g} \qquad 4.31$$

$$\overline{\bar{f}f'}=\overline{\bar{g}g'}=0 \qquad 4.32$$

$$\overline{fg}=\bar{f}\bar{g}+\overline{f'g'} \qquad 4.33$$

$$\overline{f^2}=(\bar{f})^2+\overline{(f')^2} \qquad 4.34$$

$$\overline{\left(\frac{\partial f}{\partial x}\right)}=\frac{\partial \bar{f}}{\partial x} \qquad 4.35$$

$$\frac{\partial \bar{f}}{\partial \tau}=0 \qquad 4.36$$

$$\overline{\left(\frac{\partial f}{\partial \tau}\right)}=0 \qquad 4.37$$

$$\overline{cf}=c\bar{f} \text{ (c is a constant)} \qquad 4.38$$

The following conservation equations are then obtained for steady turbulent flows:

I. Mass conservation equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0 \qquad 4.39$$

II. Momentum conservation equation:

x-component

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^2\bar{u}-\frac{\partial}{\partial x}\rho\overline{u'^2}-\frac{\partial}{\partial y}\rho\overline{u'v'}-\frac{\partial}{\partial z}\rho\overline{u'w'} \qquad 4.40$$

y-component

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^2\bar{v}-\frac{\partial}{\partial x}\rho\overline{u'v'}-\frac{\partial}{\partial y}\rho\overline{v'^2}-\frac{\partial}{\partial z}\rho\overline{v'w'} \qquad 4.41$$

z-component

$$\rho\left(\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right)=-\frac{\partial\bar{p}}{\partial z}+\mu\nabla^2\bar{w}-\frac{\partial}{\partial x}\rho\overline{u'w'}-\frac{\partial}{\partial y}\rho\overline{v'w'}-\frac{\partial}{\partial z}\rho\overline{w'^2} \qquad 4.42$$

III. Energy conservation equation

$$\rho c_p\left(\bar{u}\frac{\partial\bar{t}}{\partial x}+\bar{v}\frac{\partial\bar{t}}{\partial y}+\bar{w}\frac{\partial\bar{t}}{\partial z}\right)=k\nabla^2 t-\frac{\partial}{\partial x}\rho c_p\overline{u't'}-\frac{\partial}{\partial y}\rho c_p\overline{v't'}-\frac{\partial}{\partial z}\rho c_p\overline{w't'} \qquad 4.43$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad 4.44$$

Examination of the about equations shows that the usual steady equations (Eq. 4.16 through 4.20 with $\frac{\partial u}{\partial \tau}, \frac{\partial v}{\partial \tau}, \frac{\partial w}{\partial \tau}$ and $\frac{\partial t}{\partial \tau}$ terms equal zero) may be applied to the mean flow provided certain additional terms are included. These terms, indicated by dashed underlines and they are associated with the turbulent fluctuations. The fluctuating terms appearing in Eq. 4.40 to 4.42 represent the components of the "turbulent momentum flux" and they are usually referred to as additional "apparent stresses" or "Reynold stresses" resulting from the turbulent fluctuations. The fluctuating terms appearing in Eq. 4.43 represent the components of "the turbulent energy flux". A discussion of the meaning of "apparent stresses" and "turbulent energy flux" will be done during the study of a flow over a heated wall.

## 4.2 The Concept of Boundary Layer

Let us consider the flow of viscous fluid over a plate, as illustrated in Fig. 4.7.

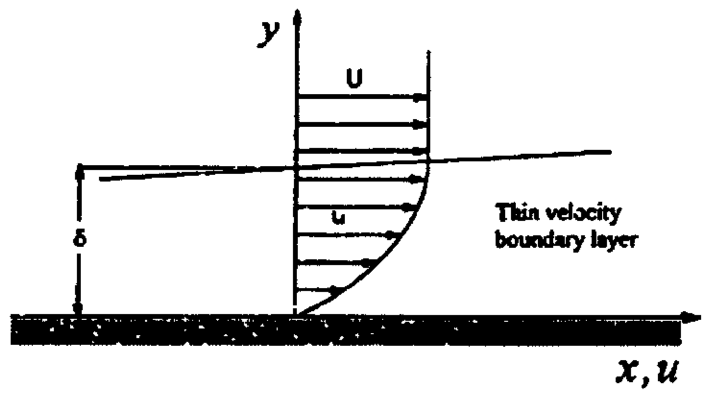



Figure 4.7 Velocity profile in the vicinity of a plate

The velocity of the fluid far from the plate (free steam velocity) is U. If the basic "non slip" assumption is made then the fluid particles adjacent to the surface adhere to it and have zero velocity. Therefore, the velocity of fluid close to the plate varies from zero at the surface to that of the free stream velocity, U. Because of the velocity gradients, viscous stresses exist in this region and their magnitude increases as we get closer to the wall. The viscous stresses tend to retard the flow in the regions near to the plate.

Based on the above observations, Ludwig Prandtl, in 1904 proposed his boundary layer theory. According to Prandtl, the motion of a fluid of small viscosity and large velocity over a wall could be separated into two distinct regions:

1. A very thin layer (known as velocity boundary layer) in the immediate neighbourhood of the wall in which the flow velocity ( $u$) increases rapidly with the distance from the wall. In this layer the velocity gradients are so large that even with a small fluid viscosity, the product of the velocity gradient and the viscosity (ie. the viscous stress, $\tau$) may not be negligible.

2. A potential flow region (or potential core) outside the boundary layer were the influence of the solid wall died out and the velocity gradients are so small that the effect of the fluid viscosity, ie. the viscous stress, $\tau$, can be ignored.

A good question would be: where is the frontier between these two regions situated? The answer is that, because of the continuous decrease of the velocity as we move off the wall, it is not possible to define the limit of the boundary layer and the beginning of the potential region. In practice, the limit of the boundary layer ie., the boundary layer thickness, is taken to be the distance to the wall at which the flow velocity has reacted some arbitrary percentage of the undisturbed free stream velocity. 99% of the free stream velocity is the most often used criterion.

### 4.2.1 Laminar Boundary Layer

The flow in the boundary layer is said to be laminar where fluid particles move along the streamlines in an orderly manner. The criterion for a flow over a flat plate to be laminar is that a dimensionless quantity called Reynolds number, $\mathrm{Re}_u$, and defined as:

$$\mathrm{Re}_x = \frac{\rho U x}{\mu} \qquad 4.45$$

should be less than $5 \times 10^5$. The number is the ratio of the inertia forces to the viscous forces.

The analytical study of the boundary can be conducted by using:

1. The fluid equations given by Eqs. 4.16 through 4.20, or
2. An approximate method based on integral equations of momentum and energy.

The integral method describes, approximately, the overall behaviour of the boundary layer. The derivation of the integral fluid equation will be given in this section. Although the results obtained by the integral approach are not complete and detailed as the results that may be obtained by the application of the differential equations, this method can still be used to obtain reasonable accurate results in many situations.

#### 4.2.1.1 Conservation Equations – Local Formulation

I. Mass and Momentum Equations

As a simple example, consider the flow and heat transfer on a flat plate as illustrated in Figure 4.8.

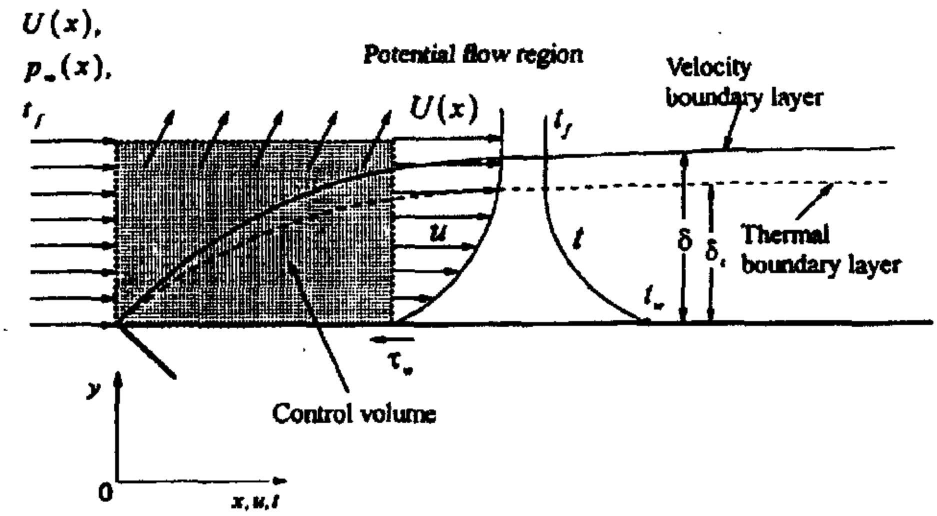



Figure 4.8 Velocity boundary layer in Laminar flow near a plane.

The x coordinate is measured parallel to the surface starting from leading edge, and the y coordinate is measured normal to it. The velocity and the pressure of the fluid far from the plate are $U(x)$ and $p_{\infty}(x)$, respectively; usually they are constant. The leading edge of the plate is sharp enough not to disturb the fluid flowing in the close vicinity of the plate. The boundary layer starts with the leading edge of the plate and the thickness is a function of the coordinate $x$. The thickness of the boundary layer is denoted by $\delta(x)$.

Assuming that the flow field is steady and two dimensional (ie., no velocity and temperature gradients in $z$ direction which is perpendicular to the plane of the sketch), body forces $\rho g_x$ and $\rho g_y$ are negligible compared to the other terms, Eqs 4.16 through 4.19 become:

Mass conservation equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad 4.46$$

Momentum conservation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial \rho}{\partial z}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad 4.47$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial \rho}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \qquad 4.48$$

The coefficent $\nu$ (m$^2$/s) is the kinematic viscosity of the fluid. Even with the above simplifications, Eqs 4.46 through 4.48 are still non-linear and cannot be solved analytically.

An order of magnitude analysis of each term of Eqs. 4.46, 4.47 and 4.48 shows that the following terms (Schlichting, 1979)

$$\nu\frac{\partial^2 u}{\partial x^2}, u\frac{\partial v}{\partial x}, v\frac{\partial v}{\partial y}, \nu\frac{\partial^2 v}{\partial x^2}\ and\ \nu\frac{\partial^2 v}{\partial y^2}\ \nu\frac{\partial^2 u}{\partial x^2}, u\frac{\partial v}{\partial x}, v\frac{\partial v}{\partial y}, \nu\frac{\partial^2 v}{\partial x^2}\ and\ \nu\frac{\partial^2 v}{\partial y^2}$$

are very small and can be ignored. Therefore, Eqs 4.46 through 4.48 become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad 4.49$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \qquad 4.50$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \qquad 4.51$$

Eq. 4.51 shows that, at a given $x$, the pressure is constant in the $y$ -direction, ie., it is independent of $y$. This result implies that the pressure gradient,

$\partial p / \partial x$ $\partial p / \partial x$, in the boundary layer, taken in a direction parallel to the wall, is equal to the pressure gradient potential flow taken in the same direction. The boundary conditions that apply to Eqs. 4.49 and 4.50 are:

At $y = 0$ $\quad u = v = 0$ 4.52

At $y = \infty$ $\quad u = U(x)$ which is usually constant 4.53

The solution of Eqs. 4.49 and 4.50 under the above boundary conditions yields the velocity distribution and the boundary layer thickness. However, this solution is beyond the scope of this course; it can be found in ScHlichting (1979) at pages 135-140.

At the outer edge of the velocity boundary layer the component of the velocity parallel to the plate, $u$, becomes equal to that of the potential flow, $U(x)$ Since there is no velocity gradient in this region;

$$\frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial^2 u}{\partial y^2} = 0 \qquad 4.54$$

and Eq. 4.50 becomes

$$U(x)\frac{\partial U(x)}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad 4.55$$

Integrating the above equation we obtain:

$$p + \frac{1}{2}\rho U^2(x) = \text{ Constant} \qquad 4.56$$

This is nothing else but the Bernoulli equation.

II. Energy conservation equation

If the temperature of the plate ($t_w$) is different from the temperature of the mainstream ($t_\infty$), a thermal boundary layer of thickness $\delta_t$ forms. Through the layer of the fluid temperature makes the transition from the wall temperature to the free stream temperature. The thickness of the thermal boundary layer is in the same order of magnitude as the velocity boundary layer thickness defined above. However, the thicknesses of both boundary layers are not necessarily equal (Figure 4.8).

For a two dimensional flow where viscous terms have been neglected in comparison with the heat added from the wall, the energy equation given by Eq. 4.20 for steady state conditions takes the following form

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}\right) \qquad 4.57$$

An order of magnitude analysis shows that the term

$$\frac{\partial^2 t}{\partial x^2}$$

Is very small and can be ignored compared to $\frac{\partial^2 t}{\partial y^2}$. Therefore Eq. 4.55 becomes:

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y^2}=\alpha\frac{\partial^2 t}{\partial y^2} \qquad 4.58$$

where $\alpha=\frac{k_f}{\rho c_p}$. For a constant temperature wall, $t_w$, the applicable boundary conditions are:

$$y=0 \quad t=t_w \qquad 4.59$$

$$y=\infty \quad t=t_\infty \qquad 4.60$$

$$x=0 \quad t=t_\infty \qquad 4.61$$

The solution of Eq 4.58 in conjunction with Eqs 4.49 and 4.50 subject to boundary conditions given by Eqs 4.52, 4.53 and 4.59 through 4.61 yields the temperature distribution and the thickness of the thermal boundary layer defined with the same criterion as the velocity boundary layer. The knowledge of the temperature distribution in the thermal boundary layer allows us to determine in conjunction with Eq 4.2 the convection heat transfer coefficient, $h$.

### 4.2.1.2 Conservation Equations – Integral Formulation

One of the important aspects of boundary layer theory is the determination of the shear forces acting on a body and the convective heat transfer coefficient if the temperature of the wall is different from that of the free stream. As was discussed in the previous section, such results can be obtained from the governing equation for laminar boundary layer flow. We also pointed out that the solutions of these equations were quite difficult and were not within the scope of the present course. In this section, we will discuss an alternative method called "integral method" to analyze the boundary layer and to determine the shear stress and the convection heat transfer coefficient. The use of this method simplifies greatly the mathematical manipulations and the results agree reasonably well with the results of exact solutions.

The integral method of analyzing boundary layers, introduced by Von Karman (1946), consists of fixing the attention on the overall behaviour of the layer as far as the conservation of mass, momentum and energy principles are concerned rather than on the local behaviour of the boundary layer.

In the derivation of the integral boundary layer equation, the integral conservation equations given in Chapter 2, Eqs 2.21, 2.22, and 2.33 will be used. These equations for a fixed control volume (ie., $\vec{w}=0$) and under steady state flow conditions have the following forms:

Mass conservation:

$$\int_A \vec{n}\cdot\rho\vec{v}\,dA = 0 \qquad 4.62$$

Momentum conservation (volume forces (i.e. gravity) are neglected:

$$-\int_A \vec{n}\cdot\rho\vec{v}\vec{v}\,dA - \int_A \vec{n}\cdot p\bar{\bar{I}}\,dA + \int_A \vec{n}\bar{\bar{\sigma}}\,dA = 0 \qquad 4.63$$

Energy equation (enthalpy form):

$$\int_A \vec{n}\cdot\rho h\vec{v}\,dA + \int_A \vec{n}\cdot\vec{q}''\,dA = 0 \qquad 4.64$$

where the kinetic and potential energies as well as the viscous dissipation are neglected; there are also no internal sources. $U$ is the internal energy.

I. Boundary layer mass conservation equation

Consider again the flow on a flat plate illustrated in Fig 4.9. As already discussed, a boundary layer develops over the plate and its thickness increases in some manner with increasing distance $x$. For the analysis, let us select a control volume (Fig 4.9) bounded by the two planes $ab$ and $cd$ which are perpendicular to the wall and a distance $dx$ apart, the surface of the plate, and a parallel plane in the free stream at a distance $\ell$ from the wall.

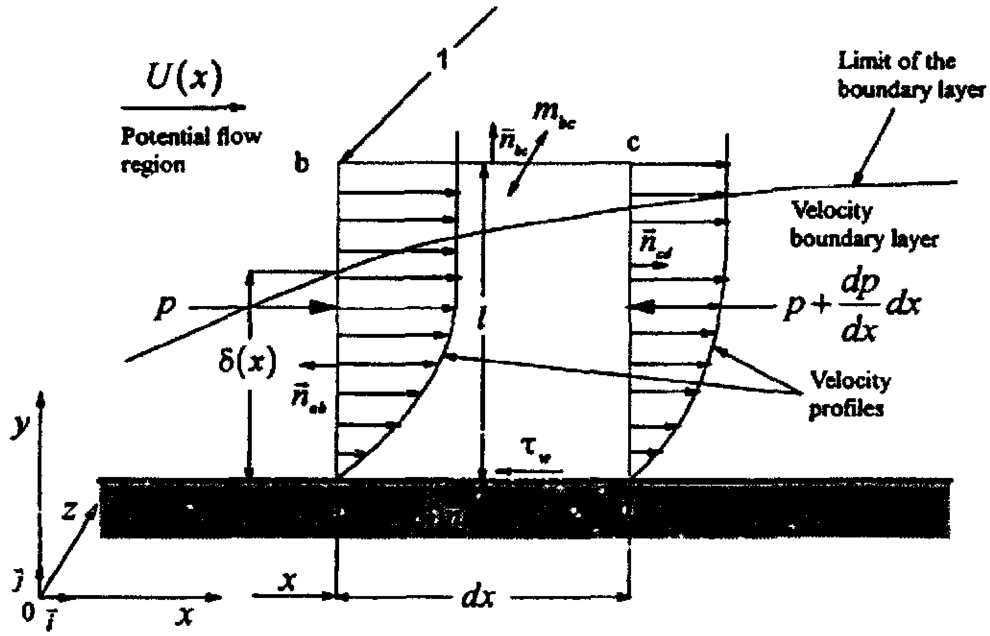



Figure 4.9 Control volume for approximate analysis of boundary layer

Assuming that the flow is steady and incompressible, the application of Eq. 4.62 to the control volume yields:

$$\int_A \vec{n}\cdot\rho\vec{v}dA = \dot{m}_{ab} + \dot{m}_{dc} + \dot{m}_{ad} = 0 \qquad 4.65$$

where $\dot{m}$ is the mass flow rate entering or leaving the control volume:

$$\dot{m}_{ab} = \int_{A_{ab}} \vec{n}_{ab} \cdot u\vec{i}\,\rho dA = \left[\int_0^\ell \rho u dy\right]_x \qquad 4.66$$

where, for a unit width of the plate $A = \ell x 1$ and $dA = 1 \cdot dy$:

$$\dot{m}_{cd} = \int_{A_{cd}} \vec{n}_{bc} \cdot u\vec{i}\,\rho dA = \left[\int_0^\ell \rho u dy\right]_{x+dx} \qquad 4.67$$

$A_{ab} = A_{cd} = \ell x1$. A Taylor expansion of Eq. 4.67 allows us to write:

$$\dot{m}_{cd} = \left[\int_0^\ell \rho u dy\right]_x + \frac{d}{dx}\left[\int_0^\ell \rho u dy\right]_x dx \qquad 4.68$$

In this expansion only the first two terms are considered, since the terms of higher order are small and can be neglected compared to the first two.

$$\dot{m}_{ad} = 0, \text{solid wall} \qquad 4.69$$

Substituting Eqs. 4.66, 4.68 and 4.69 into Eq. 4.65 we obtain for $\dot{m}_{bc}$:

$$\dot{m}_{bc} = -\frac{d}{dx}\left[\int_0^\ell \rho u dy\right]_x dx \qquad 4.70$$

II. Momentum conservation equation

In the derivation of the integral momentum conservation equation we will assume that there is no pressure variation in the direction perpendicular to the plate, the viscosity is constant and stress forces acting on all faces except the face $ad$ are negligible. Applying the momentum conservation equation (Eq. 4.63) to the control volume in Fig. 4.9 and indicating by $M$ the momentum, we write:

$$-M_{ab} - M_{cd} - M_{bc} - \int_{A_{ab}} \vec{n}_{ab} \cdot p_x \bar{I} dA - \int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} \bar{I} dA + \int_{A_{ad}} \vec{n}_{ad} \cdot \bar{\bar{\sigma}} dA = 0 \qquad 4.71$$

$$M_{ab} = \int_{A_{ab}} \vec{n}_{ab} \cdot \rho(u\vec{i})(u\vec{i}) dA = \left[\int_0^\ell \vec{n}_{ac} \rho u^2 dy\right]_x = \vec{n}_{ab}\left[\int_0^\ell \rho u^2 dy\right]_x \qquad 4.72$$

$$M_{cd} = \int_{A_{cd}} \vec{n}_{cd} \cdot \rho(u\vec{i})(u\vec{i}) dA = \left[\int_0^\ell \vec{n}_{cd} \rho u^2 dy\right]_{x+dx}$$

$$= \vec{n}_{cd}\left[\int_0^\ell \rho u^2 dy\right]_{x+dx} = \vec{n}_{cd}\left[\int_0^\ell \rho u^2 dy\right]_{x+dx} \qquad 4.73$$

A Taylor expansion allows us to write:

$$M_{cd} = \vec{n}_{cd}\left[\int_0^{\ell} \rho u^2 dy\right]_x + \vec{n}_{cd}\frac{d}{dx}\left[\int_0^{\ell} \rho u^2 dy\right]_x dx \qquad 4.74$$

$$M_{bc} = m_{bc}\vec{i} \cdot U(x) = -U(x)\vec{i}\frac{d}{dx}\left[\int_0^{\ell} \rho u dy\right]_x dx \qquad 4.75$$

The forces acting on the control volume consist of pressure and viscous stress forces:

Pressure force acting on surface $ab$:

$$\int_{A_{ab}} \vec{n}_{ab} \cdot p_x \bar{\bar{I}} dA = \vec{n}_{ab} \cdot p_x \ell \qquad 4.76$$

and on surface $cd$:

$$\int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} dA = \vec{n}_{cd}\, p_{x+dx} \ell \qquad 4.77$$

Knowing that

$$p_{x+dx} = p_x + \frac{dp_x}{dx} dx$$

and substituting it into Eq. 4.76, we obtain:

$$\int_{A_{cd}} \vec{n}_{cd} \cdot p_{x+dx} dy = \vec{n}_{cd}\left(p_x + \frac{dp_x}{dx} dx\right)\ell \qquad 4.78$$

Viscous stress forces acting on the surface <u>ad</u>:

$$\int_{A_{ad}} \vec{n}_{ad} \cdot \bar{\bar{\sigma}} dA = \vec{n}_{ad} \cdot \bar{\bar{\sigma}} dx \qquad 4.79$$

since the flow is two dimensional, the stress tensor consists of:

$$\bar{\bar{\sigma}} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix}$$

and

$$\vec{n}_{ad}\bar{\bar{\sigma}} = \begin{vmatrix} 0 \\ -1 \end{vmatrix} \cdot \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} = \begin{vmatrix} -\sigma_{yx} \\ -\sigma_{yy} \end{vmatrix} \qquad 4.80$$

$\sigma_{yx}$ is nothing else but $\tau_w$ and $\sigma_{yy}$ is negligible. Therefore, Eq. 4.78 becomes:

$$\int_A \vec{n}_{ad} \bar{\bar{\sigma}} dA = -\tau_w \vec{i} dx \qquad 4.81$$

The stresses acting on surface $ab$ and $cd$ are negligible. Since $\ell$ is chosen so that $bc$ lies outside of the boundary layer, there is no viscous stress acting on that face.

Substituting 4.72, 4.74, 4.76, 4.78, and 4.81 into 4.71 we obtain:

$$-\vec{n}_{ab}\left[\int_0^\ell \rho u^2 dy\right]_x - \vec{n}_{cd}\left[\int_0^\ell \rho u^2 dy\right]_x - \vec{n}_{cd}\frac{d}{dx}\left[\int_0^\ell \rho u^2 dy\right]_x dx$$

$$+U(x)\vec{i}\frac{d}{dx}\left[\int_0^\ell \rho u dy\right]dx - \vec{n}_{ab}p_x\ell - \vec{n}_{cd}p_x l - \vec{n}_{cd}\frac{dp_x}{dx}\ell dx - \tau_w\vec{i}dx = 0 \qquad 4.82$$

Multiplying Eq. 4.82 by $\vec{i}$ and knowing that:

$\vec{n}_{ab}\cdot\vec{i} = -1,\ \vec{n}_{cd}\cdot\vec{i} = 1$, we obtain:

$$-\frac{d}{dx}\int_0^\ell \rho u^2 dy + U(x)\frac{d}{dx}\int_0^\ell \rho u dy - \frac{dp}{dx}\ell - \tau_w = 0 \qquad 4.83$$

or

$$-\frac{d}{dx}\int_0^\ell \rho u^2 dy + U(x)\frac{d}{dx}\int_0^\ell \rho u dy = \frac{dp}{dx}\ell + \tau_w \qquad 4.84$$

Adding and subtracting to the left hand side of Eq. 4.84 the term:

$$\frac{dU(x)}{dx}\left[\int_0^\ell \rho u dy\right]$$

and with some algebra, we obtain for a constant $\rho$ the following equation:

$$\rho\frac{d}{dx}\int_0^\ell (U(x)-u)u dy - \rho\frac{dU(x)}{dx}\int_0^\ell u dy = \ell\frac{dp}{dx} + \tau_w \qquad 4.85$$

Using Eq. 4.56, $\frac{dp}{dx}$ can be written as

$$\frac{dp}{dx} = \rho U(x)\frac{dU(x)}{dx} \qquad 4.86$$

and knowing that the pressure in $y$-direction is constant, the term $\ell\frac{dp}{dx}$ is written as:

$$\ell \frac{dp}{dx} = \int_0^\ell \frac{dp}{dx} dy = -\int_0^\ell \rho U(x) \frac{dU(x)}{dx} dy \qquad 4.87$$

Combining Eq. 4.85 and 4.87 we obtain:

$$\rho \frac{d}{dx} \int_0^{\delta(x)} \left(U(x) - u\right) u dy + \rho \frac{dU(x)}{dx} \int_0^{\delta(x)} \left(U(x) - u\right) dy = \tau_w \qquad 4.88$$

In this equation $\ell$ is set equal to the boundary layer thickness, $\delta(x)$ since both integrals on the left hand side of Eq. 4.88 are zero for $y > \delta(x)$. Eq. 4.88 is the "integral momentum equation" of a steady, laminar and incompressible boundary layer. If the velocity distribution is known, then the integrands of the two integrals are known, and $\tau_w$ may be easily determined:

$$\tau_w = \mu \left( \frac{du}{dy} \right)_{y=0}$$

The resulting expression may then be interpreted as a differential equation for $\delta(x)$, the boundary layer thickness, as a function of $x$. Eq. 4.88 will be used later to determine $\delta(x)$.

III. Energy conservation equation

The integral energy equation may be obtained in a similar way to that of the momentum equation. As already discussed, the thermal boundary layer thickness, $\delta_t(x)$, is defined as the distance from the heat exchange wall at which the fluid temperature reaches 99% of its uniform value in the potential flow region. The thickness of the thermal boundary layer has the same order of magnitude as the velocity boundary layer. However, the thicknesses of these two layers are not necessarily equal. Figure 4.10 shows a fluid flowing over a constant temperature wall $t_S$. The temperature of the potential flow is $t_f$. We assume that $t_f > t_s$, although the reverse may also be true. Once more the control volume consists of two planes perpendicular to the wall and a distance $dx$ apart, the surface of the plate and a plane taken outside of both boundary layers (Figure 4.10). In the derivation of the integral energy equation, Eq. 4.54 will be used. Note the $h$ in Eq. 4.64 is the enthalpy.

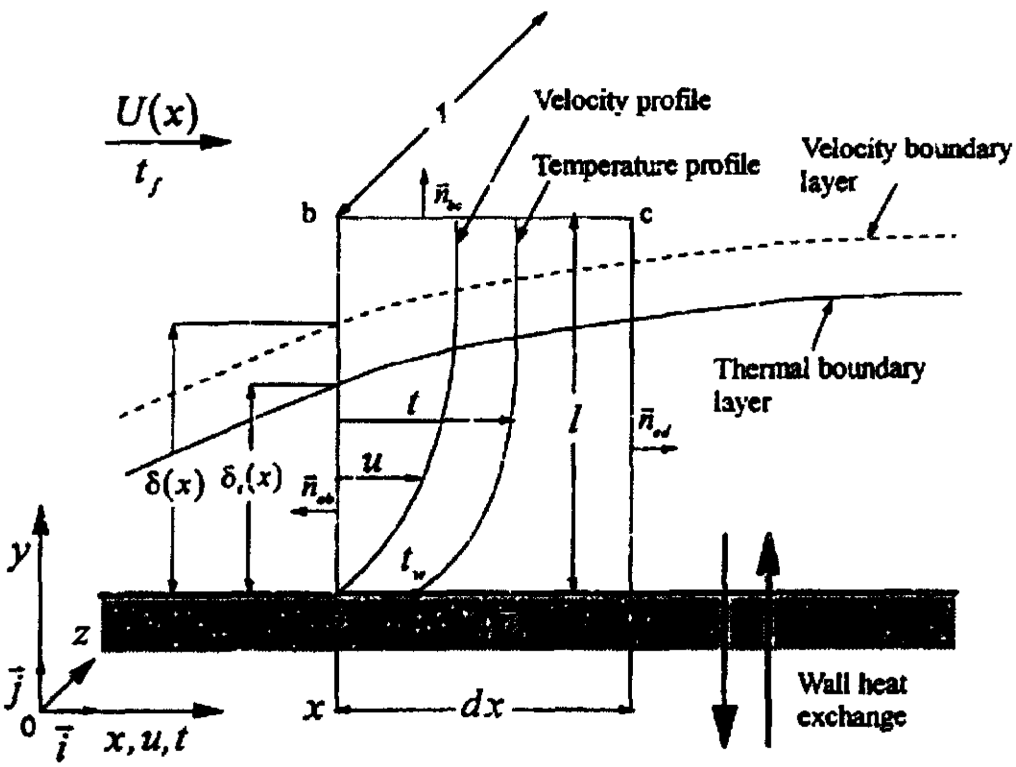



Figure 4.10 Control volume for integral conservation of energy

In the present derivation, we will assume that the kinetic and potential energies and viscous dissipation are small compared to the other quantities and thus they are neglected. Applying the energy conservation equation (Eq. 4.64) to the control volume seen in Fig. 4.9 and indicating by $E$ the flow energy, we write:

$$E_{ab} + E_{cd} + E_{bc} + \int_{A_{ad}} \vec{n}_{ad} \cdot \vec{q}''dA = 0 \tag{4.89}$$

where:

$$E_{ab} = \int_{A_{ab}} \vec{n}_{ab}\rho\left(u\vec{i}\right)hdf = -\left[\int_0^{\ell} \rho uhdy\right]_x \tag{4.90}$$

$$E_{cd} = \int_{A_{cd}} \vec{n}_{cd}\rho\left(u\vec{i}\right)hdA = \left[\int_0^{\ell} \rho uhdy\right]_{x+dx}$$

$$= \left[\int_0^{\ell} \rho uhdy\right]_x + \frac{d}{dx}\left[\int_0^{\ell} \rho uhdy\right]_x dx \tag{4.91}$$

$$E_{bc} = -\frac{d}{dx}\left[\int_0^{\ell} \rho u h_f dy\right]_x dx \qquad 4.92$$

$$\int_{A_{ad}} \vec{n}_{ad} \cdot \vec{q}'' dA = \vec{n}_{ad} \cdot \vec{q}'' dx = \vec{n}_{ad} \cdot \left[-k_f \frac{dt}{dy}\bigg|_{y=0} \vec{j}\right] dx = k_f \frac{dt}{dy}\bigg|_{y=0} dx \qquad 4.93$$

Substituting Eqs. 4.90 through 4.93 into 4.89, we obtain

$$\frac{d}{dx}\left[\int_0^{\ell} \rho u h dy\right] - \frac{d}{dx}\left[\int_0^{\ell} \rho u h_f dy\right] + k_f \frac{dt}{dy}\bigg|_{y=0} = 0 \qquad 4.94$$

or

$$\frac{d}{dx}\int_0^{\ell} \rho u (h_f - h) dy = k_f \frac{dt}{dy}\bigg|_{y=0} \qquad 4.95$$

We can also write that

$$h_f - h = c_p (t_f - t)$$

therefore Eq. 4.94 becomes

$$\frac{d}{dx}\int_0^{\delta_t(x)} c_p \rho u (t_f - t) dy = k_f \frac{dt}{dy}\bigg|_{y=0} \qquad 4.96$$

Equations 4.88 and 4.96 can also be used in the approximative analysis of steady turbulent boundary layers by replacing $u$ by $\bar{u}$ and by using an appropriate expression to describe turbulent shear stress and heat flux on the wall.

In this equation, $\ell$ is set equal to the thermal boundary layer thickness $\delta_t(x)$, since the integral on the left-hand side of Eq. 4.96 is zero for $y > \delta_t(x)$. Eq. 4.96 is the "integral energy equation" of a steady, laminar and incompressible boundary layer.

### 4.2.2 Turbulent Boundary Layer

So far we have discussed equations of conservation for a laminar boundary layer. However, in many applications, the boundary layer is turbulent. In this section we will discuss the basic features of turbulent boundary layer. In laminar flows, we have seen the heat and momentum are transferred across streamlines only by molecular diffusion and the cross flow of properties is rather small. In turbulent flow, the mixing mechanism, besides the molecular transport, consists also macroscopic transport of fluid particles from adjacent layers enhancing, therefore, the momentum and heat transport or in general, property transport.

In order to understand the basic features of the turbulent boundary layers and the governing equations we will assume that these equations for a flow over a flat plate may be obtained from laminar boundary layer equations (Eqs. 4.49, 4.50 and 4.58) by replacing $u, v, t$ by;

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$
$$t = \overline{t} + t'$$

and taking the time average of these equations. We will further assume that the velocity of the potential flow is constant. This implies that pressure also constant. Under these conditions, the conservation equations for a turbulent boundary layer are:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \qquad 4.97$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \nu \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial x}\overline{u'^2} - \frac{\partial}{\partial y}\overline{u'v'} \qquad 4.98$$

$$\overline{u}\frac{\partial \overline{t}}{\partial x} + \overline{v}\frac{\partial \overline{t}}{\partial y} = \alpha \frac{\partial^2 \overline{t}}{\partial y^2} - \frac{\partial}{\partial x}\overline{u't'} - \frac{\partial}{\partial y}\overline{v't'} \qquad 4.99$$

Based on the experiments, in Eqs. 4.98 and 4.99 the terms $\frac{\partial}{\partial x}\left(\overline{u'^2}\right)$ can be neglected relative to $\frac{\partial}{\partial y}\left(\overline{u'v'}\right)$ and $\frac{\partial}{\partial x}\overline{u't'}$ relative to $\frac{\partial}{\partial y}\overline{v't'}$. Consequently, the momentum and energy equations become:

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\mu\frac{\partial \overline{u}}{\partial y} - \frac{\partial}{\partial y}\overline{u'v'}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial y}\tau_\ell - \frac{\partial}{\partial y}\overline{u'v'} \qquad 4.100$$

$$\bar{u}\frac{\partial \bar{t}}{\partial x} + \bar{v}\frac{\partial \bar{t}}{\partial y} = \alpha\frac{\partial^2 \bar{t}}{\partial y^2} - \frac{\partial}{\partial y}\overline{v't'}$$

$$= -\frac{1}{\rho c_p}\frac{\partial}{\partial y}q_\ell'' - \frac{\partial}{\partial y}\overline{v't'} \qquad 4.101$$

In the above equations the laminar expression for shear and heat flux

$$\tau_\ell = \mu\frac{\partial \bar{u}}{\partial y}$$

$$q'' = -k\frac{\partial \bar{t}}{\partial y}$$

have been introduced to emphasize the origin of the terms involved. The shear stress $\tau_\ell$ and heat flux $q_\ell''$ represent the flux of momentum and energy in the $y$ direction due to molecular scale activity.

To understand the significance of $\frac{\partial}{\partial y}\overline{u'v'}$ and $\frac{\partial}{\partial y}\overline{v't'}$ terms, let us consider a two dimensional flow in which the mean value of the velocity is parallel to the x-direction as illustrated in Figure 4.11. Because of the turbulent nature of the flow, at a given point, the instantaneous velocity of the fluid change continuously in direction and magnitude as illustrated in Figure 4.12. The instantaneous velocity components for the present flow are:

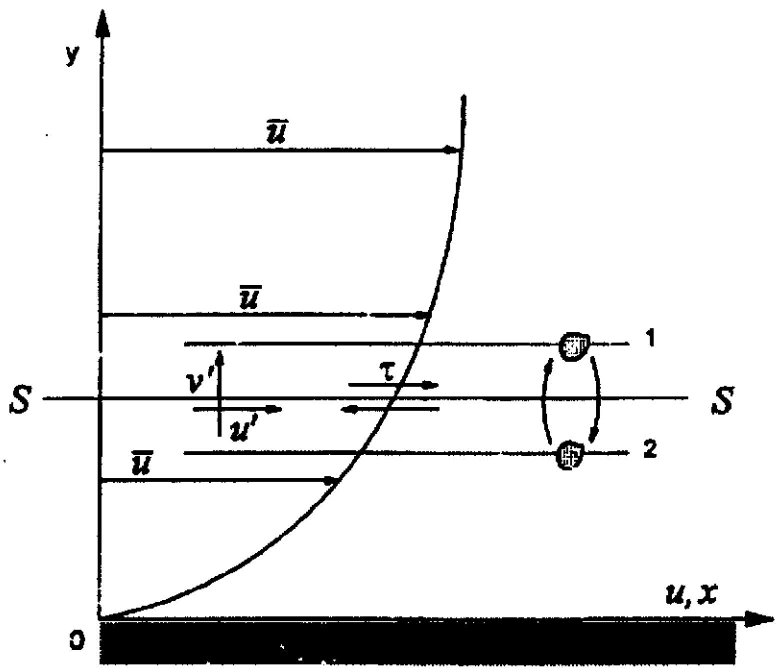



Figure 4.11 Turbulent momentum exchange

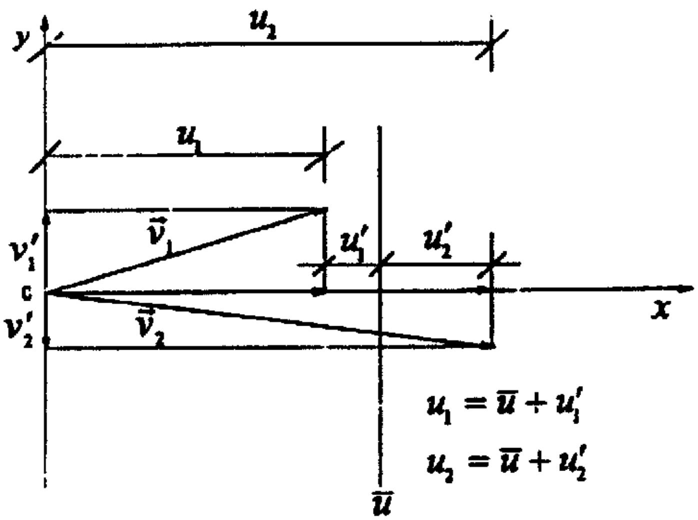



Figure 4.12 Instantaneous turbulent velocities

$$u = u + u' \qquad 4.102$$

$$v = v' \qquad 4.103$$

As discussed in Section 4.1, an exchange of molecules between the fluid layers on either side of the plane $SS$ will produce a change in the $x$-direction momentum of the fluid because of the existence of the gradient in the $x$-direction velocity. This momentum change of shearing force in the fluid which is directed in the $x$-direction denoted by $\tau_\ell$ in Eq. 4.10. If turbulent velocity fluctuations occur both in the $x$ and $y$ directions, which in the case under study, the $y$-direction fluctuations, $v'$ transport fluid lumps which are large in comparison to molecular transport across the surface $SS'$ as illustrated in Fig. 4.11. For a unit area of $SS$, the instantaneous rate of mass transport across $SS$ is:

$$\rho v' \tag{4.104}$$

This mass transfer is accompanied by a transport of $x$-direction momentum. Therefore, the instantaneous rate of transfer in the $y$-direction of $x$-direction momentum per unit area is given by:

$$-\rho v'\left(\bar{u}+u'\right) \tag{4.105}$$

where the minus sign, as will be shown later, takes into account the statistical correlation between $u'$ and $v'$. The time average of the $x$-momentum transfer gives rise to a turbulent shear stress or Reynolds stress:

$$\tau_t = -\frac{1}{\Delta\tau_0}\int_{\Delta\tau_0} \rho v'\left(\bar{u}+u'\right)d\tau \tag{4.106}$$

Breaking up Eq. 4.106 into two parts, the time average of the first is:

$$\frac{1}{\Delta\tau_0}\int_{\Delta\tau_0} (\rho v')\bar{u}\,d\tau = 0 \tag{4.107}$$

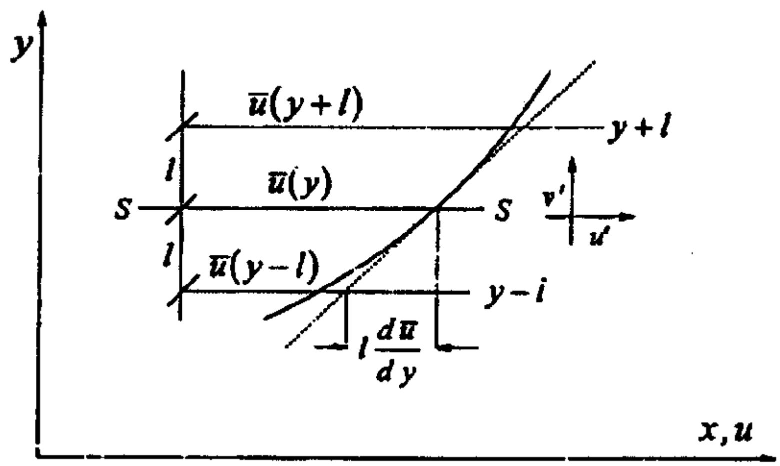
since $\bar{u}$ is a constant and the time average of $(\rho v')$ is zero. Integrating the second term of Eq. 4.106 gives:

$$\tau_t = -\frac{1}{\Delta\tau_0}\int_{\Delta\tau_0} (\rho v')u'\,d\tau = -\overline{(\rho v')u'} \tag{4.108}$$

or if $\rho$ is constant:

$$\tau_t = -\rho \overline{v'u'} \qquad 4.109$$

where $\overline{v'u'}$ is time average of the product of $u'$and $v'$. We must note that even though $\overline{u'} = \overline{v'} = 0$, the average of the fluctuation product $\overline{u'v'}$ is not zero. To understand the reàson for introducing a minus sign in Eq. 4.105, let us consider Fig. 4.13. From this figure we can see that the fluid lumps which travel upward $(v' > 0)$ arrive at a layer in the fluid in which the mean




4.13 Mixing length for momentum transfer

velocity $\bar{u}$ is larger than the layer from which they come. Assuming that the fluid particles keep on the average their original velocity $\bar{u}$ during their migration, they will tend to slow down other fluid particles after they have reached their destination and thereby give rise to a negative component $u'$. Conversely, if $v'$ is negative, the observed value of $u'$ at the new destination will be positive. On the average, therefore, a positive $v'$ is associated with a negative $u'$, and vice versa. The time average of $u'v'$ is therefore not zero but a negative quantity. The turbulent shear stress defined by Eq. 4.109 is thus positive and has the same sign as the corresponding laminar shear stress $\tau_\ell$. Based on the above discussion Eq. 4.100 can be written as:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\tau_\ell + \tau_t\right) \qquad 4.110$$

$$\tau = \tau_\ell + \tau_t \qquad 4.111$$

is called total shear stress in turbulent flow.

To relate the turbulent momentum transport to the time-average velocity gradient $d\bar{u}/dy$, Prandtl postulated that (Schlichting, 1979) the macroscopic transport of momentum (also heat) in turbulent flow is similar to that of molecular transport in laminar flow. To analyze molecular momentum transport (Section 4.1) we have introduced the concept of mean free path, or the average distance a molecule travels between collisions. Prandtl used the same concept in the description of turbulent momentum transport and defined a mixing length (known as Prandtl mixing length), $\ell$ which shows the distance travelled by fluid lumps in a direction normal to the mean flow while maintaining their identity and physical properties. (e.g., momentum parallel to $x$-direction). Referring again to Fig. 4.13, consider a fluid lump located at a distance $\ell$ (Prandtl mixing length) above and below the surface $SS$. The velocity of the lump at $y+\ell$ would be:

$$\bar{u}(y+\ell) \cong \bar{u}(y) + \ell \frac{\partial \bar{u}}{\partial y} \tag{4.112}$$

while at $y-\ell$:

$$\bar{u}(y-\ell) \cong \bar{u}(y) - \ell \frac{\partial \bar{u}}{\partial y} \tag{4.113}$$

If a fluid lump moves from layer $y-\ell$ to the layer $y$ under the influence of a positive $v'$, its velocity in the new layer will be smaller than the velocity prevailing there of an amount:

$$\bar{u}(y-\ell) - \bar{u}(y) \cong -\ell \frac{\partial \bar{u}}{\partial y} \tag{4.114}$$

Similarly, a lump of fluid which arrives at $y$ from the plane $y+\ell$ possesses a velocity which exceeds that around it, the difference being:

$$\bar{u}(y+\ell) - \bar{u}(y) = \ell \frac{\partial \bar{u}}{\partial y} \tag{4.115}$$

Here $v' < 0$. The velocity differences caused by the transverse motion can be regarded as the turbulent velocity components at $y$. Examining Eqs. 4.114 and 4.115, it can be concluded that the $u'$ fluctuation is in the same order of magnitude of $\ell \frac{\partial u}{\partial y}$, i.e.,

$$u' \cong \ell \frac{\partial \bar{u}}{\partial y} \qquad 4.116$$

Substituting Eq. 4.116 into 4.109, we obtain

$$\tau_t = -\rho \overline{v'\ell} \frac{\partial \bar{u}}{\partial y} \qquad 4.117$$

or calling $\varepsilon_m = -\overline{v'\ell}$, apparent kinematic viscosity,

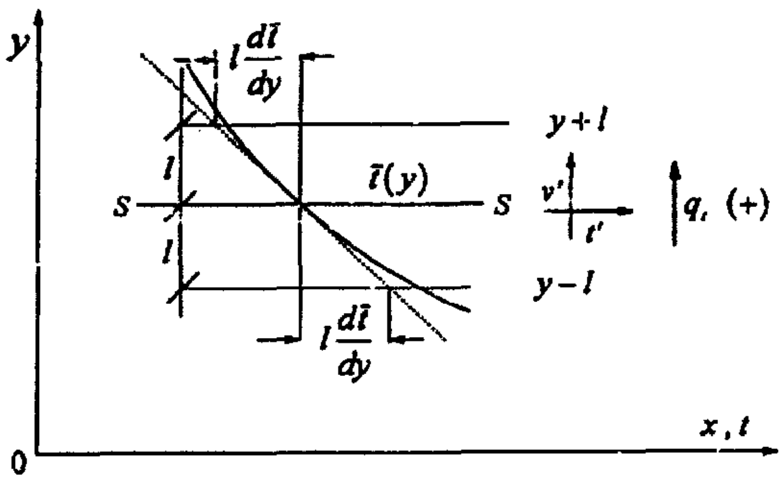
$$\tau_t = \rho \varepsilon_m \frac{\partial \bar{u}}{\partial y} \qquad 4.118$$

The total shear stress is than given by (Eq. 4.111)

$$\tau = \tau_\ell + \tau_t (\nu + \varepsilon_m) \frac{\partial \bar{u}}{\partial y} \qquad 4.119$$

Prandtl has also argued that the $v'$ fluctuation is of the same order as $u'$. $\varepsilon_m$, the apparent kinematic viscosity, is not a physical property of the fluid as $\mu$ or $\nu$; it depends on the motion of the fluid and also on many parameters; the most important is the Reynolds number of the flow. $\varepsilon$ is also known to vary from point to point in the flow field; it vanishes near the solid boundary where the transverse fluctuations disappear. The ratio of $\varepsilon / \nu$ under certain circumstances can go as high as 400 to 500. Under such cases, the viscous shear (i.e., $\nu$) is negligible in comparison to the turbulent shear $(\varepsilon_m)$ and may be omitted.

The transfer of heat in a turbulent flow can be modelled in a similar way to that of the momentum transfer. Let us consider a two-dimensional time-mean temperature distribution shown in Fig 4.14. The fluctuating velocity $v'$ continuously transports fluid particles and the energy stored in them across the surface $SS$.




4.14 Mixing length for energy transfer in turbulent flow

The instantaneous rate of energy transfer per unit have at any point in the $y$ direction is:

$$\rho v'(c_p t) \qquad 4.120$$

where $t = \bar{t} + t'$. The time average of the turbulent heat transfer is given by:

$$q_t'' = \frac{1}{\Delta\tau_0}\int_{\Delta\tau_0} \rho v'(\bar{t} + t')d\tau \qquad 4.121$$

Carrying out the above integration we obtain:

$$q_t'' = \rho c_p \overline{v't'} \qquad 4.122$$

Substituting Eq. 4.122 into 4.101 we obtain:

$$\bar{u}\frac{\partial \bar{t}}{\partial x} + \bar{v}\frac{\partial \bar{t}}{\partial y} = -\frac{1}{\rho c_p}\frac{\partial}{\partial y}(q_\ell'' + q_t'') \qquad 4.123$$

$$q'' = q_\ell'' + q_t'' \qquad 4.124$$

is called total heat flux in turbulent flow.

Using Prandtl's concept of mixing length we can write that:

$$t' \cong \ell\frac{d\bar{t}}{dy} \qquad 4.125$$

Combining Eqs. 4.122 and 4.125 we obtain:

$$q_t'' = \rho c_p \overline{v't'} = -c_p \rho \overline{v'\ell} \frac{d\bar{t}}{dy} \qquad 4.126$$

Here, it is assumed that the transport mechanism of energy and momentum are similar; therefore, the mixing lengths are equal. The product $\overline{v't'}$ is positive on the average because a positive $v'$ is accomplished by a positive $t'$ and vice versa. The minus sign which appears in Eq. 4.17 is the consequence of the conversion that the heat is taken to be positive in the direction of increasing $y$; this also ensures that heat flows in the direction of decreasing temperature, thus satisfies the second law of thermodynamics.

Representing by $\varepsilon_h = \overline{v'\ell}$ Eq. 4.126 becomes

$$q_t' c_p \rho \varepsilon_h \frac{d\bar{t}}{dy} \qquad 4.127$$

The total heat transfer is then given by (4.124)

$$q'' = -(k + c_p \rho \varepsilon_h)\frac{d\bar{t}}{dy} \qquad 4.128$$

or

$$q'' = -c_p \rho (\alpha + \varepsilon_h)\frac{dt}{dy} \qquad 4.129$$

where $\alpha = k / \rho c_p$ is the molecular diffusivity of heat. $\varepsilon_h$ is called the "eddy diffusivity of heat" or eddy heat conductivity.

## 4.3 Forced Convection Over a Flat Plate

In the previous section we discussed velocity and thermal boundary layers with extent of $\delta(x)$ and $\delta_t(x)$, respectively. The velocity boundary layer is characterized by the presence of velocity gradients, i.e., shear stresses whereas the thermal boundary layer is characterized by temperature gradients, i.e., heat transfer. From an engineering point of view we are mainly interested in determining wall friction and heat transfer coefficients. In this section, we will focus our attention on the determination of these coefficients for laminar and turbulent flows. In order to reach rapidly the objectives, the integral momentum and energy equations (Eqs. 4.88 and 4.96) will be used.

### 4.3.1 Laminar Boundary Layer

In laminar boundary layers, we discussed that fluid motion is very orderly and it is possible to identify streamlines along which fluid particles move. Fluid motion along a streamline has velocity components in $x$ and $y$ direction ( $u$ and $v$ ). The velocity component $v$ normal to the wall, contributes significantly to momentum and energy transfer through the boundary. Fluid motion normal to the plate is brought about by the boundary layer growth in the $x$-direction (Figure 4.8)

Consider a flat plate of constant temperature placed parallel to the incident flow as shown in Fig. 4.15. We will assume that the potential velocity $U(x)$ and temperature $t_f$ are constant. The constant potential velocity, according Eq. 4.56 implies that longitudinal pressure gradient (in either the potential region or on the boundary layer) is zero. It will also be assumed that

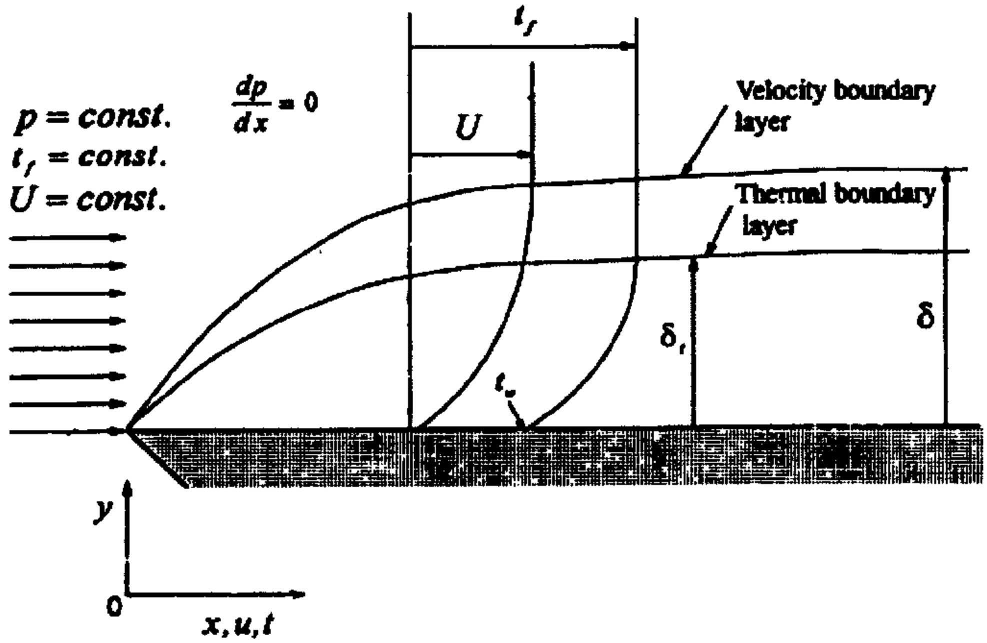



Figure 4.15 Velocity & thermal boundary layers for laminar flow past a flat plate

all physical properties are independent of temperature. In the following we will determine the friction and heat transfer coefficient on the plate.

#### 4.8.1.1 Velocity Boundary Layer – Friction Coefficient

For flow past a flat plate in which $U(x) = U$ (a constant) and $dp/dx = 0$, the integral momentum equation (Eq. 4.88) becomes:

$$\rho \frac{d}{dx}\int_0^{\delta(x)} (U-u)u\,dy = \tau_w \qquad 4.130$$

The above momentum integral equation involves two unknown velocity component $u(x,y)$ and the boundary layer thickness. If the velocity profile were known, it is then possible to obtain an expression for the boundary layer thickness. A typical velocity profile in the boundary layer is sketched in Fig. 4.15. This profile can be represented with a third degree polynomial of $y$ in the form:

$$u(x,y) = a(x) + b(x)y + c(x)y^2 + d(x)y^3 \qquad 4.131$$

This polynomial must satisfy the following boundary conditions:

$$y = 0 \qquad u = 0 \qquad 4.132$$

$$y = \delta \qquad u = U \qquad 4.133$$

$$y = \delta \qquad \frac{\partial u}{\partial y} = 0 \qquad 4.134$$

$$y = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0 \qquad 4.135$$

The last condition is obtained for a constant pressure condition from Eq. 4.50 by setting the velocities $u$ and $v$ equal to zero at $y = 0$. We will also assume that the velocity profiles at various $x$ positions are similar; i.e., they have the same functional dependence on the $y$ coordinate. Using the boundary conditions 4.132 through 4.135, $a$, $b$, $c$ and $d$ are determined as:

$$a=0 \quad b=\frac{3}{2}\frac{U}{\delta} \quad c=0 \quad d=-\frac{1}{2}\frac{U}{\delta^3} \tag{4.136}$$

and the following expression is obtained for the velocity profile:

$$\frac{u}{U}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{4.137}$$

Substituting Eq. 4.137 into Eq. 4.130, knowing that:

$$\tau_w=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\frac{3}{2}\mu\frac{U}{\delta} \tag{4.138}$$

and carrying out the integration we obtain:

$$\frac{39}{280}\rho U^2\frac{d\delta}{dx}=\frac{3}{2}\mu\frac{U}{\delta} \tag{4.139}$$

or

$$\delta d\delta=\frac{140}{13}\frac{\mu}{\rho}\frac{1}{U}dx \tag{4.140}$$

The integration of the above equation yields:

$$\delta=4.64\sqrt{-\frac{\mu}{\rho}\frac{1}{U}x+const.} \tag{4.141}$$

Referring to Fig. 4.15 we see that at $x=0,\ \delta=0$, therefore the constant is zero and the variation of the velocity boundary layer thickness is given by:

$$\delta=4.64\sqrt{\frac{\mu}{\rho}\frac{x}{U}} \tag{4.142}$$

or

$$\frac{\delta}{x}=\frac{4.64}{\sqrt{\frac{\rho U_m}{\mu}}}=\frac{4.64}{\mathrm{Re}_x^{1/2}} \tag{4.143}$$

where $R_x=\frac{\rho U_x}{\mu}$ is the Reynolds number based on $x$, distance from the leading edge.

The exact solution of the boundary layer equations (Eqs. 4.49 and 4.50) yields:

$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}} \qquad 4.144$$

Therefore, Eq. 4.28 yields a value for $\delta(x)$ 8% less than that of the exact analysis. Since most of the experimental measurements are only accurate to within 10%, the results of the approximate analysis are satisfactory in practice.

Combining Eqs. 4.138 and 4.143, we obtain for wall shear stress:

$$\tau_w = 0.323 \frac{\rho U^2}{\sqrt{\mathrm{Re}_x}} \qquad 4.145$$

Wall friction coefficient is defined as :

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} \qquad 4.146$$

or with Eq. 4.145

$$C_f = \frac{0.646}{\sqrt{\mathrm{Re}_x}} \qquad 4.147$$

This is the local friction coefficient. The average friction coefficient is given by:

$$C_x = \frac{\int_0^x C_f dx}{x} \qquad 4.148$$

or with Eq. 4.147

$$C_x = \frac{1.292}{\sqrt{\mathrm{Re}_x}} \qquad 4.149$$

**4.3.1.2 Thermal Boundary Layer – Heat Transfer coefficient**

Now, we will focus our attention to the thermal boundary layer and determine the heat transfer coefficient. Consider again the flow on a flat plate illustrated in Fig. 4.15. The temperature of the plate is kept at $t_w$ starting from the leading edge and the temperature of the potential flow is $t_p$ and is constant. Under the above conditions, a thermal boundary layer starts forming at the leading edge of the plate. In order to determine the heat transfer coefficient, the thickness of the thermal boundary layer, $\delta_t(x)$, should be known. To do so, we will use the integral energy equation given by Eq. 4.96 and rewritten here for convenience:

$$\frac{d}{dx}\int_0^{\delta_t(x)} c_p \rho u\left(t_f - t\right)dy = k_f \left.\frac{dt}{dy}\right|_{y=0} \qquad 4.96$$

The above integral equation involves two unknowns: the temperature $t(x,y)$ and the thermal boundary layer thickness. If the temperature profile were known it would then be possible to obtain an expression for the thermal boundary layer. The velocity profile has already been determined and given by Eq. 4.137. A typical temperature profile is given in Fig. 4.15. The profile, in a way similar to that of the velocity profile, can be represented with a third degree polynomial of $y$ in the form:

$$t(x,y) = a(x) + b(x)y + c(x)y^2 + d(x)y^3 \qquad 4.150$$

with boundary conditions:

$$y = 0 \qquad t = t_w \qquad 4.151$$

$$y = \delta_t \qquad t = t_f \qquad 4.152$$

$$y = \delta_t \qquad \frac{\partial t}{\partial y} = 0 \qquad 4.153$$

$$y = 0 \qquad \frac{\partial^2 t}{\partial y^2} = u \qquad 4.154$$

The last condition is obtained from Eq. 4.58 by setting the velocities $u$ and $v$ equal to zero at $y = 0$. Under these conditions the temperature distributions is given by:

$$\frac{t - t_w}{t_f - t_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad 4.155$$

Introducing a new temperature defined as:

$$\theta = t - t_w \qquad 4.156$$

Eqs. 4.96 and 4.155 can be written as:

$$\frac{d}{dx}\int_0^{\delta_t} c_p\rho u(\theta_w-\theta)dy = k_f\left.\frac{d\theta}{dy}\right|_{y=0} \qquad 4.157$$

and

$$\frac{\theta}{\theta_w}=\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad 4.158$$

where $\theta_w = t_f - t_w$

Substituting in Eq. 4.157 $u$ and $\theta$ by Eq. 4.137 and 4.158, we obtain:

$$\theta_w U\frac{d}{dx}\int_0^{\delta_t}\left[1-\frac{3}{2}\frac{y}{\delta_t}+\frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\left[\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy=\frac{3}{2}\alpha\frac{\theta_w}{\delta_t} \qquad 4.159$$

where $\alpha = k_f / \rho c_p$

Defining $\xi$ as the ratio of the thermal boundary layer thickness to the velocity boundary layer thickness $\delta_t/\delta$, introducing this new parameter into Eq. 4.159, performing the necessary algebraic manipulations and carrying out the integration we obtain:

$$\frac{d}{dx}\left[\delta\left(\frac{3}{20}\xi^2-\frac{3}{280}\xi^4\right)\right]=\frac{3}{2}\frac{\alpha}{\xi\delta}\frac{1}{U} \qquad 4.160$$

A simplification can be introduced at this point if we accept the fact that $\xi$, the ratio of boundary layer thickness, will be near 1 or better less than unity. We will later see that this is true for Prandtl (Pr) numbers equal or greater than 1. This situation is met for a great number of fluids. With the above assumption the second term in the bracket in Eq. 4.160 can be neglected compared to the first one and this equation becomes:

$$\frac{3}{20}\frac{d}{dx}\delta\xi^2=\frac{3}{2}\frac{\alpha}{\xi\delta}\frac{1}{U} \qquad 4.161$$

or

$$\frac{1}{10}\left(\xi^3\delta\frac{d\delta}{dx}+2\xi^2\delta^2\frac{d\xi}{dx}\right)=\frac{\alpha}{U} \qquad 4.162$$

Taking into account Eq. 4.140:

$$\delta\frac{d\delta}{dx}=\frac{140}{13}\frac{\mu}{\rho}\frac{1}{U} \qquad 4.140$$

and Eq. 4.142

$$\delta = 4.64\sqrt{\frac{\mu}{\rho}\frac{x}{U}} \qquad 4.142$$

and introducing them into Eq. 4.162, we obtain:

$$\frac{14}{13}\frac{\mu}{\rho}\frac{1}{\alpha}\left(\xi^3 + 4x\xi^2\frac{d\xi}{dx}\right) = 1 \qquad 4.163$$

The ratio $\frac{\mu}{\rho\alpha}$ or $\frac{\nu}{\alpha}$ is a non-dimensional quantity frequently used in heat transfer calculations; it is called the Prandtl number and has the following form:

$$\Pr = \frac{\mu}{\rho}\frac{1}{\alpha} = \frac{\mu}{\rho}\frac{\rho c_p}{k_f} = \frac{c_p\mu}{k_f} \qquad 4.164$$

With the above definition, Eq. 4.163 becomes;

$$\left(\xi^3 + 4x\xi^2\frac{d\xi}{dx}\right) = \frac{13}{14}\frac{1}{\Pr} \qquad 4.165$$

or

$$\left(\xi^3 + \frac{4}{3}x\frac{d}{dx}\xi^3\right) = \frac{13}{14}\frac{1}{\Pr} \qquad 4.166$$

Making a change of variable as :

$$y = \xi^3$$

Eq 3.166 is written as:

$$y + \frac{4}{3}x\frac{dy}{dx} = \frac{13}{14}\frac{1}{\Pr} \qquad 4.167$$

The homogeneous and particular solutions of the above equations are:

Homogeneous $\quad y = cx^n$ with $n = -3/4$ $\qquad$ 4.168

Particular $\quad y = \frac{13}{14}\frac{1}{\Pr}$ $\qquad$ 4.169

The general solution is then given by:

$$y = \frac{13}{14}\frac{1}{\Pr} + Cx^{-\frac{3}{4}} \qquad 4.170$$

or

$$\xi^3 = \frac{13}{14}\frac{1}{\mathrm{Pr}} + Cx^{-\frac{3}{4}} \qquad 4.171$$

Since the plate is heated starting from the leading edge, the constant C must be zero to avoid an indeterminate solution at the leading edge, therefore:

$$\xi^3 = \frac{13}{14}\frac{1}{\mathrm{Pr}} \qquad 4.172$$

or

$$\xi = \frac{1}{1.026\ \mathrm{Pr}^{1/3}} \qquad 4.173$$

In the forgoing analysis the assumption was made that $\xi \le 1$. This assumption, according Eq. 4.173 is satisfactory for fluids having Prandtl numbers greater than about 0.7. For a Prandtl number equal to 0.7, $\xi$ is about 0.91 which is close enough to 1 and the approximations we made above is still acceptable. Fortunately, most gases and liquids have Prandtl numbers higher than 0.7. Liquid metals constitute an exception: their Prandtl numbers are in the order of 0.01. Consequently the above analysis cannot be applied to liquid metals.

Returning now to our analysis, we know that the local heat transfer coefficient was given by Eq. 4.2 which, for the present case, is written as:

$$h_x = \frac{-k_f\left(\frac{\partial t}{\partial y}\right)_{y=0}}{t_w - t_f} \qquad 4.174$$

or in terms of $\theta(= t - t_w)$ and $\theta_w(= t_f - t_w)$

$$h_x = \frac{k_f\left(\frac{\partial \theta}{\partial y}\right)_{y=0}}{\theta_w} \qquad 4.175$$

where

$$\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = \frac{3}{2}\frac{\theta_w}{\delta_t} = \frac{3}{2}\frac{\theta_w}{\xi^{\delta}} \qquad 4.176$$

Substituting Eq. 4.176 in Eq. 4.175, we obtain:

$$h_x = \frac{3}{2}\frac{k_f}{\xi\delta} \qquad 4.177$$

Combining the above equation with Eqs. 4.173 and 4.142 we obtain:

$$h_x = 0.332 k_f \sqrt[3]{\Pr}\sqrt{\frac{\rho U}{\mu x}} \qquad 4.178$$

This equation may be made non-dimensional by multiplying both sides by $x / k_f$:

$$\frac{h_x x}{k_f} = 0.332\sqrt[3]{\Pr}\sqrt{\frac{\rho U_x}{\mu}} \qquad 4.179$$

or

$$Nu_x = 0.332\sqrt[3]{\Pr}\sqrt{\mathrm{Re}_x} \qquad 4.180$$

where $Nu_x$ is the Nusselt number, defined as:

$$Nu_x = \frac{h_x x}{k_f} \qquad 4.181$$

Eq. 4.180 express the local value of the heat transfer coefficient in terms of the distance from the leading edge of the plate, potential flow velocity and physical properties of the fluid. The average heat transfer coefficient can be obtained by integrating over the length of the plate:

$$\bar{h} = \frac{\int_0^L h_x dx}{\int_0^L dx} = 2h_{x=L} \qquad 4.182$$

$$\overline{Nu_L} = \frac{\bar{h}L}{k_f} = 2Nu_{x=L} \qquad 4.183$$

or

$$\overline{Nu_L} = \frac{\bar{h}L}{k_f} = 0.664\,\mathrm{Re}_L^{1/2}\Pr^{1/3} \qquad 4.184$$

where

$$\mathrm{Re}_L = \frac{\rho U L}{\mu} \qquad 4.185$$

The above analysis was based on the assumption that the fluid properties were constant throughout the flow. If there is a substantial difference between the wall and free stream temperature, the fluid properties should be evaluated at the mean film temperature defined as:

$$t_m = \frac{t_w + t_f}{2} \qquad 4.186$$

It should be pointed out that the expressions given by Eqs. 4.180 and 4.185 apply to a constant temperature and when:

$$\Pr \geq 0.7$$
$$\mathrm{Re}_x \leq 5\text{x } 10^5$$

However, in many practical problems the surface heat flux is constant and the objective is to find the distribution of the plate surface temperature for given fluid flow conditions. For the constant heat flow case it was shown that the local Nusselt number is given by:

$$Nu_x = 0.453 \mathrm{Re}_x^{1/2} \Pr^{1/3} \qquad 4.187$$

### 4.3.2 Turbulent Boundary Layer

A turbulent boundary layer is characterized by velocity fluctuations. These fluctuations enhance considerably the momentum and energy transfer, i.e., increase surface friction as well as convection heat transfer. The turbulent boundary layer doesn't start developing with the leading edge of the plate. The development of the velocity boundary on a flat plate is sketched in Fig 4.16. The boundary layer is initially laminar. At some distance from the leading edge, the laminar flow in the boundary layer becomes unstable and a gradual transition to turbulent flow occurs. The length over which this transition takes

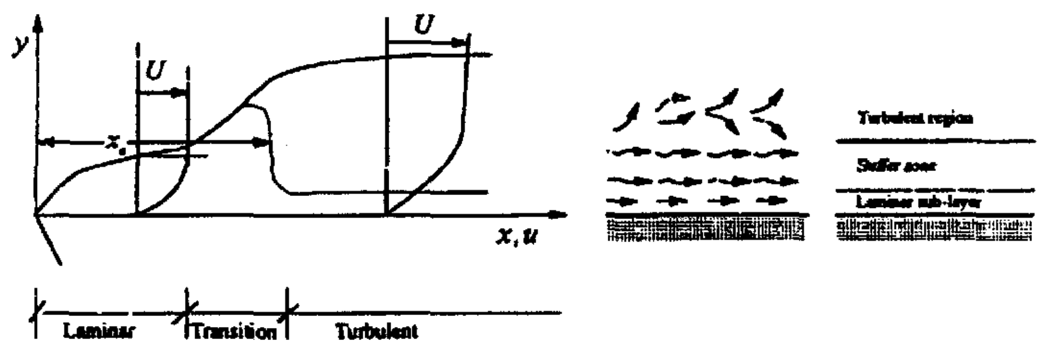



Figure 4.16 Development of laminar & turbulent boundary layers on a flat plate

place is called " the transition zone". In the fully turbulent region, flow conditions are characterized by a highly random, three-dimensional motion of fluid lumps. The transition to turbulence is accompanied by an increase of the boundary layer thickness, wall shear stress and the convection heat transfer coefficient.

In the turbulent boundary layer three different regions are observed (Fig. 4.16):

1. A laminar sub-layer in which the diffusion dominates the property transport, and the velocity and temperature profiles are nearly linear.
2. A buffer zone where the molecular diffusion and turbulent mixing are comparable to property transport.
3. A turbulent zone in which the property transport is dominated by turbulent mixing.

An important point in engineering applications is the estimation of the transition location from laminar to turbulent boundary layer. This location, denoted by $x_c$, is tied to a dimensionless grouping of parameters called the Reynolds number:

$$\mathrm{Re}_x = \frac{\rho U_x}{\mu} \qquad 4.188$$

where the characteristic length $x$ is the distance from the leading edge. The critical Reynolds number is the value of $\mathrm{Re}_x$ for which the transition to turbulent boundary layer begins. For a flow over a flat plate the critical Reynolds number, depending on the roughness of the surface and the turbulence level of the free steam, varies between $10^5$ to $3\text{x}10^6$. It is usually recommended to use a value of $\mathrm{Re}_x = 5\text{x}10^5$ for the transition point from laminar to turbulent flow. From Fig. 4.16, it is obvious that the transition zone for a certain length and a single point of transition is an approximation. However, for most engineering applications this approximation is acceptable. We should also emphasize that the flow in the transition zone is quite complex and our knowledge of it is very limited.

### 4.3.2.1 Velocity Boundary Layer – Friction Coefficient

Analytical treatment of a turbulent boundary layer is very complex. This is due to the fact that the apparent kinematic viscosity $\varepsilon_m$, as already pointed out in Section 4.2.2, is not a property of the fluid but depends on the motion of the fluid itself, the boundary conditions, etc. Here, we will use a simple approach to determine the thickness of the turbulent boundary layer.

The general characteristics of a turbulent boundary layer are similar to those of a laminar boundary layer: the time-average flow velocity varies rapidly from zero at the wall to the uniform value of the potential core. Due to the transverse turbulent fluctuations, the velocity distribution is much more curved near the wall than in a laminar flow case. However, the same distribution at the outer edge of the turbulent layer is more uniform than that corresponding to a laminar flow.

A number of experimental investigations have shown that the velocity in a turbulent boundary layer may be adequately described by a one-seventh power law:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad 4.189$$

where $\delta$ is the boundary layer thickness and $u$ is the time average of turbulent velocity. For the sake of simplicity the bar notation to show the time average is dropped with understanding that all turbulent velocities referred to are time-averaged velocities. This power law represents well the experimental velocity profiles for local Reynolds numbers in the range $5 \times 10^5 < \mathrm{Re}_x < 10^7$. Although Eq. 4.189 describes well the velocity distribution in the turbulent layer, it is not valid at the surface. This can be seen when we try to evaluate the shear stress on the wall which has the following form:

$$\tau \approx \left.\frac{du}{dy}\right|_{y=0} \qquad 4.190$$

According to Eq. 4.189, the velocity gradient anywhere in the boundary is:

$$\frac{du}{dy} = \frac{1}{7} \frac{U}{\delta^{1/7} y^{6/7}} \qquad 4.191$$

This relation leads to an infinite value of the stress at the wall. This is not physically possible. In our previous discussion, we have pointed out that the turbulence dies out in the vicinity of the wall and the boundary layer behaves in a laminar fashion. In this region, the velocity distribution is assumed to be linear. Based on the above discussion, the velocity distribution in the turbulent region, including the buffer zone, will follow one-seventh power law whereas in the laminar sub-layer will be linear. This linear variation will be selected so that the slope at $y = 0$ yields the wall stress obtained experimentally by Blasius (1913) for turbulent flows on a smooth plate:

$$\tau_w = 0.0228 \rho U^2 \left( \frac{\nu}{U\delta} \right)^{1/4} \qquad 4.192$$

where $\nu$ is the kinematic viscosity. The velocity distribution in the laminar sub-layer will join to that in the fully turbulent region at a distance $\delta_S$ from the wall. This distance is commonly called the thickness of the laminar sub-layer. The resulting velocity profile is sketched in Fig. 4.17.

To determine the turbulent boundary layer thickness, we will employ Eq. 4.130, repeated here for convenience $(U = \text{constant},\ dp/dx = 0)$:

$$\rho \frac{d}{dx} \int_0^{\delta(x)} (U - u) u dy = \tau_w \qquad 4.130$$

Although the above equation is derived for a laminar flow, it may also be used for a turbulent flow as long as the velocities used are time average velocities and as long as wall shear term, $\tau_w$, is adequately represented for turbulent flow, for example with Eq. 4.192 for a turbulent flow on a plate. For the purposes of integral analysis, the momentum integral in Eq 4.130 can be evaluated by using the power law given by Eq. 4.189. This is justified by the fact that the laminar sub-layer is very thin. For the wall shear stress, the Blasius correlation given by

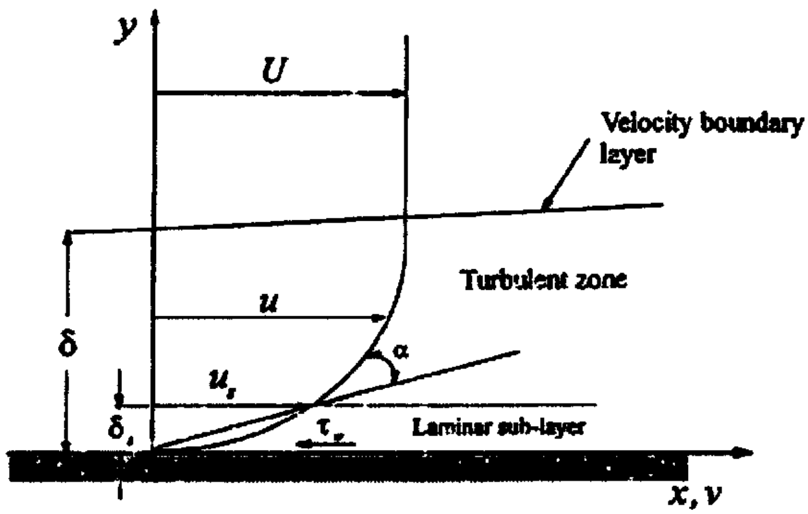



Figure 4.17 Velocity profiles in the turbulent zone and laminar sub-layer

Eq. 4.192 will be used. Consequently, the substitution of Eqs. 4.189 and 4.192 into 4.130 yields:

$$\rho U^2 \frac{d}{dx}\int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right] = 0.0228\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4} \qquad 4.193$$

Integration of the above equation yields:

$$\frac{7}{72}\frac{d\delta}{dx} = 0.0228\left(\frac{\nu}{U\delta}\right)^{1/4}$$

or

$$\delta^{1/4}\, d\delta = 0.235\left(\frac{\nu}{U}\right)^{1/4} dx \qquad 4.194$$

This equation can be easily integrated to obtain:

$$\delta = 0.376\left(\frac{\nu}{U}\right)^{1/5} x^{4/5} + \text{constant} \qquad 4.195$$

The constant may be evaluated for two physical situations:

1. The boundary layer is fully turbulent from the leading edge of the plate. In this case boundary condition will be:

$$x = 0 \quad \delta = 0 \qquad 4.196$$

2. The boundary layer follows a laminar growth pattern up to $R_c = 5 \times 10^5$ and a turbulent growth thereafter. In this case the boundary condition will be:

$$\delta = \delta_\ell \text{ at } x = x_{cr} = 5 \times 10^5 \frac{\mu}{\rho U} \qquad 4.197$$

$\delta_\ell$ can be obtained from Eq. 4.144 repeated here for convenience:

$$\frac{\delta}{x} = \frac{4.64}{\mathrm{Re}_x^{1/2}} \qquad 4.198$$

as:

$$\delta_\ell = 4.64\, x_c \left(5 \times 10^5\right)^{-1/2} \qquad 4.199$$

For our application, we will retain the first option. Consequently, the constant in Eq. 4.195 is zero and the thickness of the boundary layer is given by:

$$\frac{\delta}{x} = \frac{0.376}{\left(\frac{\rho U x}{\mu}\right)^{1/5}} = 0.376\,\mathrm{Re}_x^{-1/5} \qquad 4.200$$

This assumption is not true in practice. However, experiments show that the predictions of Eq. 4.200 agree well with data. When Eq. 4.144 and 4.200 are compared we observe that the thickness of the turbulent boundary layer increases faster than that of laminar boundary layer. The equations for boundary layer thickness (i.e., Eqs. 4.144 and 4.200) apply only to the regions of fully laminar or fully turbulent boundary layers. As can be seen from Fig. 4.16 the transition from laminar flow to turbulent flow does not occur at a definite point, but rather occurs over a finite length of the plate. The transition zone is a region of highly irregular motion and the knowledge of the flow in this region is quite limited. For most engineering applications, it is customary to assume that the transition from laminar boundary to turbulent boundary occurs suddenly when the local $\mathrm{Re}_x$ number is equal to $5 \times 10^5$. At the transition point the thickness of the boundary layers will be different with the turbulent boundary layer thicker than the laminar boundary layer; i.e., a discontinuity exists in the thickness.

Let us now focus our attention to the laminar sub-layer, illustrated in Fig. 4.17 and determine its thickness, $\delta_S$, and the velocity of the fluid at the juncture between the laminar sub-layer and the fully turbulent zone denoted by $u_S$.

Since the velocity varies linearly in the laminar sub-layer, the shear stress in this layer is given by

$$\tau = \mu \frac{du}{dy} = \mu \frac{u}{y} \qquad 4.201$$

Combining this expression with the wall shear stress correlation established by Blasius for a turbulent boundary, we obtain for the variation of the velocity the following expression:

$$u = 0.0228 \rho \frac{U^2}{\mu} \left( \frac{\mu}{\rho U \delta} \right)^{1/4} y \qquad 4.202$$

when $y = \delta_S$, $u = u_S$, the above expression becomes:

$$\frac{\delta_S}{\delta} = \frac{1}{0.0228} \left( \frac{\mu}{\rho U \delta} \right)^{3/4} \frac{u_s}{U} \qquad 4.203$$

The velocity profile in the turbulent region was given by Eq. 4.189 which, for $y = \delta_S$ and $u = u_S$, can be written as:

$$\frac{\delta_S}{\delta} = \left( \frac{u_s}{U} \right)^7 \qquad 4.204$$

The combination of Eqs. 4.203 ans 4.204 yields:

$$\frac{u_s}{U} = 1.878 \left( \frac{\rho U \delta}{\mu} \right)^{-1/8} \qquad 4.205$$

$\delta$, the thickness of the turbulent boundary layer is given by Eq. 4.200, therefore Eq. 4.205 becomes:

$$\frac{u_s}{U} = 2.12 \left( \frac{\mu}{\rho U x} \right)^{0.1} = \frac{2.12}{\mathrm{Re}_x^{0.1}} \qquad 4.206$$

Combining Eqs. 4.204 and 4.206 we obtain:

$$\frac{\delta_s}{\delta} = \frac{194}{\mathrm{Re}_x^{0.7}} \qquad 4.207$$

The wall shear stress can also be written as:

$$\tau_w = \mu \frac{u_s}{\delta_s} \qquad 4.208$$

Using Eqs. 4.200, 4.206 and 4.207, the above equation becomes:

$$\tau_w = \rho U^2 \frac{0.0296}{\mathrm{Re}_x^{0.2}} \qquad 4.209$$

Dividing both sides of this equation by $\frac{1}{2}\rho U^2$ and using the definition of the wall friction coefficient, $c_f$, given by Eq. 4.146, we obtain:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0592}{\mathrm{Re}_x^{0.2}} \qquad 4.210$$

This is the local wall friction coefficient.

#### 4.3.2.2 Heat Transfer in the Turbulent Boundary Layer

The concepts regarding turbulent boundary layers discussed in Sections 4.2.2 and 4.3.2.1 will now be employed for the analysis of heat transfer as flat plates in turbulent flow. We will first discuss the Reynolds analogy for momentum and heat transfer. Subsequently we will discuss a more refined analogy introduced by Prandtl.

Reynolds Analogy for Laminar boundary layer

In a two-dimensional laminar boundary layer, the shear stress at a plane located at $y$ is given by:

$$\tau = \mu \frac{du}{dy} \qquad 4.211$$

The heat flux across the some plane is:

$$q'' = -k_f \frac{dt}{dy} \qquad 4.212$$

Dividing Eqs. 4.211 and 4.212 side by side, we obtain

$$\frac{q''}{\tau}=-\frac{k_f}{\mu}\frac{dt}{du} \qquad 4.213$$

This expression can also be written as:

$$\frac{q''}{\tau}=-\frac{k_f}{\mu c_p}c_p\frac{dt}{du} \qquad 4.214$$

$\mu c_p / k_p$ is the Prandtl number. Reynolds assumes that $\Pr = 1$, therefore Eq. 4.214 becomes:

$$\frac{q''}{\tau}=-c_p\frac{dt}{du} \qquad 4.215$$

Assuming that $q''/\tau$ ratio is constant and equal to the same ratio on the wall, i.e.,

$$\frac{q''}{\tau}=\frac{q''_w}{\tau_w} \qquad 4.216$$

Eq. 4.215 can be easily integrated from the wall conditions $t=t_w$, $u=c$, to the potential flow conditions $t=t_f$, $u=U$,

$$\frac{q''_w}{\tau_w}\int_0^U du=-c_p\int_{t_w}^{t_f}dt \qquad 4.217$$

on

$$\frac{q''_w}{t_w-t_f}=\frac{\tau_w c_p}{U} \qquad 4.218$$

The left hand side of Eq. 4.218 is nothing else but convection heat transfer, $h$. Therefore we write:

$$h=\frac{\tau_w c_p}{U} \qquad 4.219$$

Using the definition of wall friction coefficient

$$c_f=\frac{\tau_w}{\frac{1}{2}\rho U^2} \qquad 4.146$$

Eq. 4.219 becomes

$$h=\frac{1}{2}\rho U c_p c_f \qquad 4.220$$

Multiplying both sides of the above expressing $x$, knowing that for $\Pr = 1$, $c_p = k_f / \mu$, and using the definition of local Reynolds number

$$\mathrm{Re}_x = \frac{\rho U_x}{\mu}$$

and local Nusselt number:

$$Nu_x = \frac{hx}{k_f}$$

Eq. 4.147 is written as:

$$Nu_x = \frac{1}{2} c_{\tau} \mathrm{Re}_x \qquad 4.221$$

This is the dimensionless statement of Reynolds' analogy for laminar flow. Observe that heat transfer coefficient is related to the wall friction factor.

In Section 4.3.1.1 it was shown that for a laminar flow on a flat plate, the wall friction coefficient was given by (Eq. 4.147)

$$c_{\tau} = \frac{0.646}{\mathrm{Re}_x^{1/2}} \qquad 4.147$$

For this case Reynolds analogy (Eq. 4.221) gives:

$$Nu_x = 0.332 \mathrm{Re}_x^{1/2} \qquad 4.222$$

The result obtained from the integral momentum and energy analysis was (Eq. 4.180)

$$Nu_x = 0.332 \mathrm{Re}_x^{1/2} \Pr^{1/3} \qquad 4.180$$

For $\Pr = 1$, this result is the same as the one obtained by Reynolds' analogy. Therefore, there is an agreement between Eqs. 4.180 and 4.222. It appears that the effect of the Prandtl number differing from unity can be expressed by a factor equal to $\Pr^{1/3}$. The latter fact is sometimes applied in other cases when exact solution to the thermal boundary layer cannot be obtained, and experimental skim friction measurements are used to predict heat transfer coefficients.

Reynolds' analogy for turbulent boundary layer

In Section 4.2.2, we have seen that the total shear stress and heat flux in a turbulent boundary layer were given by Eqs. 4.119 and 4.129, respectively. These equations are repeated here for convenience:

$$\tau = \tau_\ell + \tau_t (\nu + \varepsilon_m)\frac{\partial \bar{u}}{\partial y} \qquad 4.119$$

$$q'' = q''_\ell + q''_t = -c_p \rho(\alpha + \varepsilon_H)\frac{dt}{dy} \qquad 4.129$$

where $\nu$: is the kinematic viscosity related to molecular momentum exchange (molecular diffusivity of momentum)

$\varepsilon_m$: is the apparent kinematic viscosity related to turbulent momentum exchange (eddy diffusivity of momentum)

$\alpha$ : is the molecular diffusivity of heat

$\varepsilon_H$: is the eddy diffusivity of heat

Reynolds assumed that the entire flow in the boundary layer was turbulent. This means that he neglected the existence of the viscous sub-layer and the buffer zone. Under these conditions the molecular diffusivities of momentum ($\nu$) and heat ($\alpha$) can be neglected in comparison with turbulent diffusivities ($\varepsilon_m$ and $\varepsilon_H$), i.e.,

$$\nu << \varepsilon_m \text{ and } \alpha << \varepsilon_4 \qquad 4.223$$

Moreover, Reynolds assumed that:

$$\varepsilon_m = \varepsilon_H = \varepsilon$$

Under these conditions, Eqs. 4.119 and 4.129 become:

$$\tau = \tau_t = \rho\varepsilon\frac{du}{dy} \qquad 4.224$$

$$q'' = q''_t = -\rho c_p \varepsilon\frac{dt}{dy} \qquad 4.225$$

Dividing Eqs. 4.225 and 4.224 side by side, we obtain

$$\frac{q''_t}{\tau_t} = -c_p\frac{dt}{du}$$

Comparing Eq. 4.226 with Eq. 4.215, we observe that these equations are similar provided that in laminar boundary layer the Prandtl number is equal to 1.

Comparing Eq. 4.226 with Eq. 4.215, we observe that these equations are similar provided that in laminar boundary layer the Prandtl number is equal to 1.

Prandtl's Modification to Reynolds' Analogy

Prandtl assumed that the turbulent boundary layer consisted of two layers:

1. A viscous layer where the molecular diffusivities are dominant, that is:

$$\nu >> \varepsilon_m \text{ and } \alpha >> \varepsilon_H \tag{4.227}$$

2. A turbulent zone where the turbulent diffusivities are dominant, that is:

$$\varepsilon_m >> \nu \text{ and } \varepsilon_H >> \alpha \tag{4.228}$$

He also assumed that $\varepsilon_m = \varepsilon_H = \varepsilon$. This approach implies that the Prandtl number is not necessarily equal to 1.

The variation of velocity and temperature in this two-region boundary layer is illustrated in Fig. 4.18. In the laminar sub-layer, the velocity and the temperature vary linearly: from zero to $u_S$ for velocity and from $t_w$ to $t_S$, for temperature. In the turbulent region the variation of the velocity, as discussed in Section 4.3.2.1 is given by one-seventh power law (Eq. 4.189) and varies from $u_S$ to U, velocity of the potential flow, whereas the temperature varies from $t_S$ to $t_p$, temperature of the potential flow.

Applying Eq. 4.213 to the laminar sub-layer we write:

$$\frac{q''}{\tau} du = -\frac{k_f}{\mu} dt \tag{4.229}$$

Integrating this equation between 0 and $u_S$ and between $t_w$ and $t_{S1}$ assuming that $\frac{q''}{\tau}$ ratio is constant and equal to $\frac{q''_w}{\tau_w}$ we obtain:

or

$$\frac{q''_w}{\tau_w} \int_0^{U_s} du = -\frac{k_f}{\mu} \int_{t_w}^{t_s} dt \tag{4.230}$$

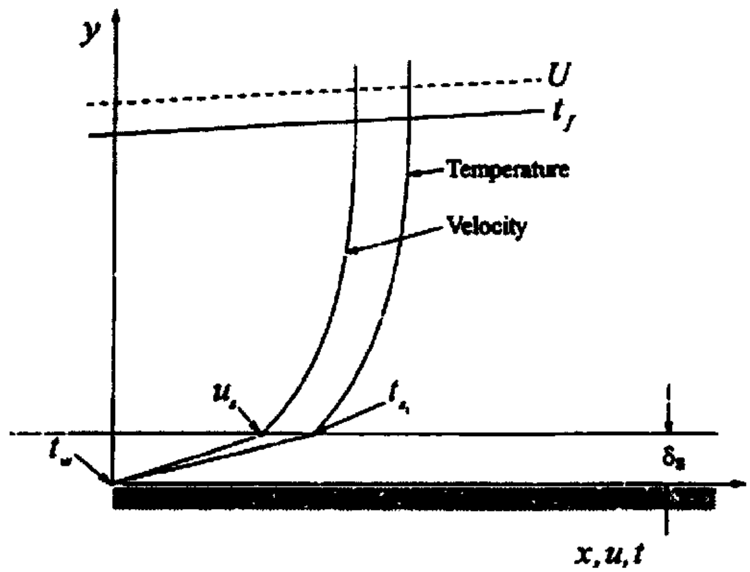



Figure 4.18 Turbulent boundary layer consisting of two layers: Prandtl approach

$$q''_w = \tau_w \frac{k_f}{\mu} \frac{1}{u_S} \left(t_w - t_{S1}\right) \qquad 4.231$$

Applying now Eq. 4.226 to the turbulent region of the boundary layer we write:

$$\frac{q''_t}{\tau_t} du = -c_p dt \qquad 4.232$$

Integrating this equation between $u_S$ and U , and between $t_s$ and $t_f$ , (see Fig. 4.18) and assuming again $q''_t/\tau_t$ ratio is constant and equal to $q''_w/\tau_w$, we obtain:

$$\frac{q''_w}{\tau_w} \int_{u_S}^{U} du = -c_p \int_{t_{s1}}^{t_f} dt \qquad 4.233$$

or

$$\frac{q''_w}{\tau_w} \int_{u_S}^{U} du = -c_p \int_{t_{s1}}^{t_f} dt \qquad 4.234$$

The elimination of $t_{s1}$ , between Equ. 4.231 and 4.234 yields

$$t_w - t_f = \frac{q''_w}{\tau_w} \left( \frac{\mu U_S}{k_f} + \frac{U - u_S}{c_p} \right) \qquad 4.235$$

Knowing that:

$$q''_w = h\left(t_w - t_f\right) \quad 4.1$$

Eq. 4.235 is written as:

$$h_c = \frac{1}{\frac{U}{\tau_w c_p}\left[\frac{c_p \mu}{k_f}\frac{u_s}{U} + \left(1 - \frac{u_S}{U}\right)\right]} \quad 4.236$$

or with $\Pr = c_p \mu / k_f$:

$$h_c = \frac{\frac{\tau_w c_p}{U}}{1 + \frac{u_s}{U}(\Pr - 1)} \quad 4.237$$

The above equation is the statement of Prandtl's modification of Reynolds' analogy and may be written in a dimensionless form by multiplying both sides $x / k_f$ and by rearranging the numerator:

$$Nu_x = \frac{\frac{1}{2}\frac{\tau_w}{\frac{1}{2}\rho U^2}\frac{\mu C_p}{k_f}\frac{Ux\rho}{\mu}}{1 + \frac{u_s}{U}(\Pr - 1)} \quad 4.238$$

Recognizing that:

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}, \Pr = \frac{\mu C_p}{k_f} \text{ and } \mathrm{Re} = \frac{Ux\rho}{\mu}$$

Eq 4.238 becomes:

$$Nu_x = \frac{\frac{1}{2}C_f \cdot \Pr \cdot \mathrm{Re}}{1 + \frac{u_s}{U}(\Pr - 1)} \quad 4.239$$

For a turbulent flow $C_f$ and $\frac{u_s}{U}$ are given with Eqs. 4.206 and 4.210, repeated here for convenience:

$$\frac{u_s}{U} = \frac{2.12}{\mathrm{Re}^{0.1}} \quad 4.206$$

$$C_f = \frac{0.0592}{\mathrm{Re}_x^{0.2}} \quad 4.207$$

Substituting these relationships into Eq. 4.239, we obtain:

$$Nu_x = \frac{0.0292\ \mathrm{Re}_x^{0.8}\ \mathrm{Pr}}{1+2.12\ \mathrm{Re}_x^{-0.1}(\mathrm{Pr}-1)} \qquad 4.240$$

This relation is found to give adequate results for turbulent heat transfer coefficients in spite of many simplifications. The fluid properties in Eq. 4.208 should be evaluated at the mean temperature and the Prandtl number should not be too different from unity. The major difficulty of Eq. 4.240 is its integration to obtain the average Nusselt number. It is observed that for $\mathrm{Pr}$ numbers not different from unity, which is the case for many gases and liquids, the denominator of Eq. 4.240 is nearly constant. Therefore, for such cases it is recommended that following expression be used in the estimation of the heat transfer coefficient:

$$Nu_x = 0.0292\ \mathrm{Re}^{0.8}\ \mathrm{Pr}^{1/3} \qquad 4.241$$

Again the mean film temperature (Eq 4.186) should be used for all properties. Eq. 4.241 can be integrated along the plate to obtain an average Nusselt number. For a plate length $L$, this average is given by

$$\bar{h}_c = \frac{1}{L}\left[0.0292\ \mathrm{Pr}^{1/3}\left(\frac{\rho U}{\mu}\right)^{0.8} k_f \int_0^L \frac{1}{x^{0.2}}\,dx\right]$$

or

$$Nu_L = \frac{\bar{h}_c L}{k_f} = 0.036\ \mathrm{Re}^{0.8}\,\mathrm{Pr}^{1/3} \qquad 4.242$$

This latter relation assumes that the boundary layer is turbulent starting from the leading edge of the plate. As discussed in Section 4.3.2 and illustrated in Fig. 4.16, starting from the leading edge over certain portion of the plate the boundary layer is laminar. The transition to turbulent flow occurs at a distance $x_c$. This point is specified by a critical Reynolds number; usually a value of $5 \times 10^5$ is used. Under these conditions, a better average film coefficient and average Nusselt number would be given by the combination of Eqs. 4.180 and 4.241:

$$\bar{h}_c = \frac{1}{L}\left[0.332\ \Pr^{1/3} k_f \left(\frac{\rho U}{\mu}\right)^{1/2} \int_0^{x_c} \frac{1}{x^{1/2}}\,dx\right.$$

$$\left. +0.0292\ \Pr^{1/3} k_f \left(\frac{\rho U}{\mu}\right)^{0.8} \int_{x_{cr}}^{L} \frac{1}{x^{0.2}}\,dx \right] \qquad 4.243$$

In the above averaging, it is assumed that the laminar to turbulent transition occurs instantaneously. The following expression is obtained for the average Nusselt number:

$$Nu_L = 0.036\ \Pr^{1/3}\left[\mathrm{Re}_L^{0.8} - \mathrm{Re}_{cr}^{0.8} + 18.44\ \mathrm{Re}_{cr}^{1/2}\right] \qquad 4.244$$

where $\mathrm{Re}_{cr}$ is the critical Reynolds number. If $\mathrm{Re}_{cr}$=500,000, the above relation becomes:

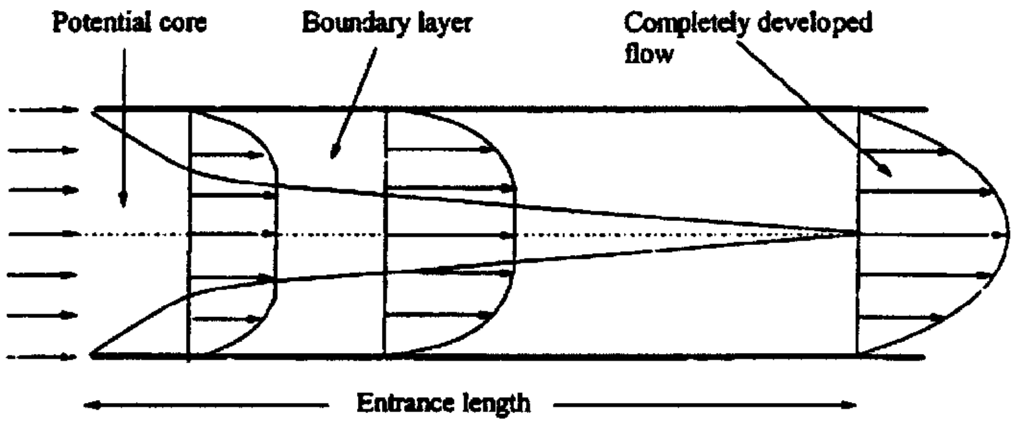
$$Nu_L = 0.036\ \Pr^{1/3}\left[\mathrm{Re}_L^{0.8} - 23{,}100\right] \qquad 4.245$$

## 4.4 Forced Convection in Ducts

Heating and cooling of fluid flowing inside a duct constitutes one of the most frequently encountered engineering problems. The design and analysis of heat exchanges, boilers, economizers, super heater and nuclear reactors depend largely on the heat exchange process between the fluid and the wall of the tubes.

The flow inside a duct can be laminar or turbulent. The turbulent flow inside ducts is the most widely encountered type in various industrial applications. Laminar flow inside ducts is mainly encountered in compact heat exchanges, in the heating and cooling of heavy fluids such as oils, etc.

When a fluid with uniform velocity enters a straight pipe a velocity boundary layer (also a thermal boundary layer, if the tube is heated) starts developing along the surface of the pipe as illustrated in Fig. 4.19.




4.19 Flow in the entrance region of a pipe

Near the entrance of the pipe, the boundary layer develops in a way similar to that on a flat plate. Because of the presence of an opposite wall on which a boundary layer also develops, at a given point along the tube, both boundary layers will touch each other and fill the entire tube. The length of the tube over which the viscous layers have grown together and filled the tube cross section is called the starting or entrance length. The flow beyond this region is termed "fully developed flow". In the entrance region the flow consist of a potential core region near the centre of the tube where the velocity is uniform and a boundary region near the wall of the tube where the velocity varies from the potential core value to zero at the wall. As we proceed along the pipe in the entrance region, the portion of the tube occupied by the boundary layer, where the flow velocity varies, increases and the position occupied by the potential core decreases. Consequently, in order to satisfy the principle of mass conservation, i.e., a constant average velocity, the velocity of the potential core shouts increase. This increase is illustrated in Fig. 4.19.

The transition from laminar to turbulent flow is likely to occur in the entrance length. If the boundary layer is laminar until it fills the tube, the flow in the fully developed region will be laminar with a parabolic velocity profile. However, if the transition from laminar to turbulent flow occurs in the entrance region, the flow in the fully developed region will be turbulent with somewhat blunter velocity profile as illustrated in Fig. 4.20. In a tube, the Reynolds number based on the diameter:

$$\mathrm{Re}_D = \frac{\rho U_m D}{\mu} \qquad 4.246$$

is used as a criterion for the transition from laminar to turbulent flow.

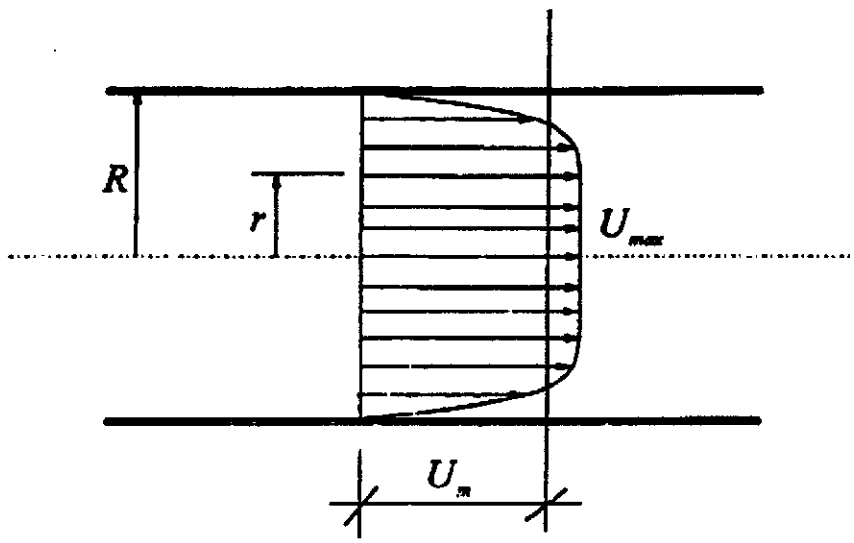



Figure 4.20 Velocity profile in turbulent pipe flow

Here $D$ is the pipe diameter and $U_m$ is the average velocity of the flow in the pipe, for:

$$\mathrm{Re}_D > 2300$$

the flow is usually observed to be turbulent. This value should not be treated as a precise value, since a range of Reynolds numbers for transition has been experimentally observed depending on the pipe roughness and the smoothness of the flow. The generally accepted range for transition is:

$$2000 < \mathrm{Re}_D < 4000 \qquad 4.247$$

For laminar flow, the length of the entrance region maybe obtained from an expression of the flow (Langhaar, 1942)

$$\frac{L_S}{D} = 0.0575\ \mathrm{Re}_D \qquad 4.248$$

There is no satisfactory expression for the entry length in turbulent flow, it is known that this length is practically independent of Reynolds number and, as a first approximation it can be assured that (Kay and Crawford, 1980.)

$$10 \le \frac{L_S}{D} \le 60 \qquad 4.249$$

Usually it is assumed that the turbulent flow is fully developed for $L_S / D > 10$.

### 4.4.1 Laminar Flow

In this section, the velocity distribution, frictional pressure loses and the convection heat transfer coefficient will be discussed for the fully developed region. The discussion of the entrance region is beyond the scope of this lecture.

#### 4.4.1.1 Velocity Distribution and Friction Factor in Laminar Flow

The form of the velocity can be easily determined for a steady state laminar flow of an incompressible, constant property flow in the fully developed region of a circular tube. In the fully developed region, velocity profile doe not change along the tube. It depends only on the radius, i.e., $u = u(r)$.

To proceed with the analysis, let us select a fix control volume of radius $r$ and length $dx$ sketched in Fig. 4.21. The application of the macroscopic momentum balance (Eq. 2.22) to the above control volume yields (see Eq. 4.63)

$$-\int_A \vec{n} \cdot \rho \vec{v}\vec{v} dA - \int_A \vec{n} \cdot p\bar{\bar{I}} dA + \int_A \vec{n} \cdot \bar{\bar{\sigma}} dA = 0 \qquad 4.63$$

Multiplying this equation by $\vec{i}$ and knowing that the velocity profile does not change in the $x$-direction $(ie., \vec{i} \int_A \vec{n} \cdot \rho \vec{v}\vec{v} dA = 0)$, we obtain:

$$-\vec{i} \cdot \int_A \vec{n} \cdot p\bar{\bar{I}} dA + \vec{i} \cdot \int_A \vec{n} \cdot \bar{\bar{\sigma}} dA = 0 \qquad 4.250$$

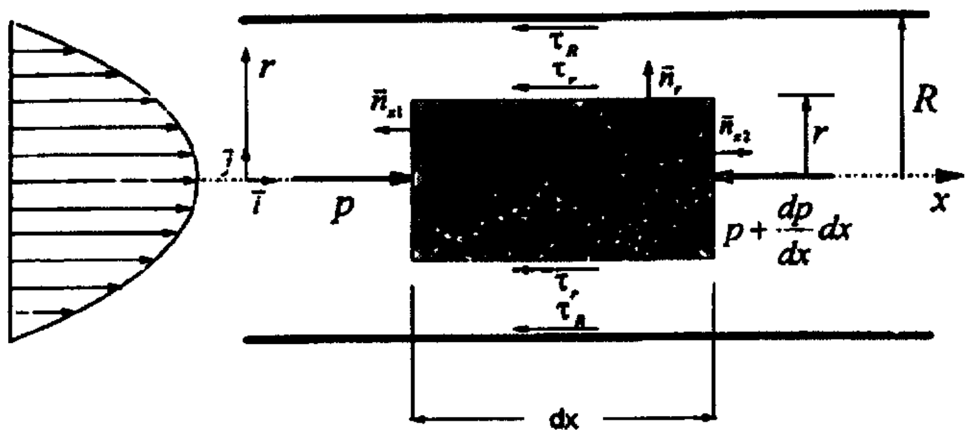



Figure 4.21 Control volume in a laminar, fully developed flow in a circular tube

which is simply a balance equation between of the forces acting on the control volume, i.e., balance between shear and pressure forces in the flow. The pressure and viscous stress terms in Eq 4.250 can be written as:

$$-\vec{i}\cdot\int_A \vec{n}\cdot p\bar{\bar{I}}\cdot dA = -p_x\pi r^2 + \left(p_x + \frac{dp}{dx}dx\right)\pi r^2 = \pi r^2\frac{dp}{dx}dx \qquad 4.251$$

$$-\vec{i}\cdot\int_A \vec{n}\cdot\bar{\bar{\sigma}}dA = -2\pi r dx\tau_r \qquad 4.252$$

Substituting Eqs. 4.251 and 4.252 into 4.250 we obtain:

$$\frac{r}{2}\frac{dp}{dx} = -\tau_r \qquad 4.253$$

We know that:

$$\tau_r = -\mu\frac{du}{dy}$$

and this equation with $y = R - r$ (or $dy = -dr$) becomes:

$$\tau_r = -\mu\frac{du}{dr} \qquad 4.254$$

Substitution of Eq. 4.254 into Eq. 4.253 yields:

$$du = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)rdr \qquad 4.255$$

Since the axial pressure gradient is independent of $r$, Eq. 4.255 may be integrated to obtain:

$$u=\frac{1}{4\mu}\left(\frac{dp}{dx}\right)r^2+C \qquad 4.256$$

The constant $C$ can be easily determined by setting:

$$r=R \quad u=0 \qquad 4.257$$

Therefore, the constant becomes:

$$C=-\frac{1}{4\mu}\left(\frac{dp}{dx}\right)R^2 \qquad 4.258$$

It follows that the velocity profile of the fully developed flow is given by:

$$u=\frac{1}{4\mu}\left(\frac{dp}{dx}\right)R^2\left(1-\frac{r^2}{R^2}\right)=U_{max}\left(1-\frac{r^2}{R^2}\right) \qquad 4.259$$

Hence the fully developed profile is parabolic. Note that the pressure gradient must always be negative.

The average velocity is given by:

$$U_m=\frac{\int_0^R 2\pi r\rho u dr}{\pi R^2} \qquad 4.260$$

Substituting $u$ in the above equation by Eq. 4.259 and carrying out the integration, we obtain:

$$U_m=-\frac{R^2}{8\mu}\left(\frac{dp}{dx}\right)=\frac{1}{2}U_{max} \qquad 4.261$$

Substituting this result into Eq. 4.259, the velocity profile is then

$$u(r)=2U_m\left(1-\frac{r^2}{R^2}\right) \qquad 4.262$$

$U_m$ can be calculated from the knowledge of the volume flow rate:

$$U_m=\frac{Q}{A} \qquad 4.263$$

where $Q$ is the volume flow rate in $m^3/s$ and $A$ is the flow section in $m^2$. Eq. 4.262 can be used to determine frictional pressure gradient.

Eq. 4.253 can also be written for a control volume bounded by the tube wall and two planes perpendicular to the axis and a distance $dx$ apart as:

$$\frac{R}{2}\frac{dp}{dx} = -\tau_R \qquad 4.264$$

or as

$$\frac{dp}{dx} = -\frac{4\tau_R}{D} \qquad 4.265$$

where $D$ is the diameter of the tube. $\tau_R$ is given by:

$$\tau_R = -\mu\left(\frac{du}{dr}\right)_{r=R} \qquad 4.266$$

Dividing both sides of Eq. 4.265 by $\frac{1}{2}\rho U_m^2$ and calling:

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2}$$

we write

$$-\frac{dp}{dx} = f \cdot \frac{1}{2}\rho U_m^2 \frac{1}{D} \qquad 4.268$$

where $f$ is the friction factor. Since the velocity distribution is known $\tau_R$ can be easily evaluated from Eq. 4.266 as:

$$\tau_R = 8\mu\frac{U_m}{D} \qquad 4.269$$

The substitution of Eq. 4.269 into Eq. 4.267

$$f = \frac{64\mu}{\rho U_m D} = \frac{64}{\mathrm{Re}_D} \qquad 4.270$$

where $\mathrm{Re}_D\left(=\frac{\rho U_m D}{\mu}\right)$ is the Reynolds number based on the diameter of the pipe.

Equation 2.268 in conjunction with Eq. 2.270 allows us to evaluate the frictional gradient component of the total pressure gradient in a laminar flow. The pressure drop in a tube of length $L$ is obtained by integrating Eq. 2.268:

$$\Delta p = -\int_{P_1}^{P_2} dp = \int_0^L f\frac{1}{2}\rho\frac{U_m^2}{D}dx = f\frac{1}{2}\rho U_m^2\frac{L}{D} \qquad 4.271$$

#### 4.4.1.2 Bulk Temperature

For flow over a flat plate, the convection heat transfer coefficient was defined as:

$$h = \frac{q''}{t_w - t_f} \qquad 4.272$$

where $t_f$ is the temperature of the potential stream. However, in a tube flow there is no easily discernible free-stream condition. Even the centerline temperature ( $t_c$) cannot be easily expressed in terms of inlet flow parameter and wall heat flux. Consequently, for fully developed pipe flow it is customary to define a so-called "bulk temperature" in the following form:

$$t_b = \frac{\int_0^R \rho c_p tu 2\pi r dr}{\int_0^R \rho u c_p 2\pi r dr} \qquad 4.273$$

For incompressible flow with constant $c_p$, this definition is written as:

$$t_b = \frac{\int_0^R \rho C_p tu 2\pi r dr}{\dot{m} C_p} \qquad 4.274$$

The numerator of Eq. 4.273 or 4.274 represents the total energy flow through the tube, and the denominator represents the product of mass flow and specific heat integrated over the flow area. The bulk temperature is thus representative of the total energy of the flow at a particular location along the tube. Consequently, as can be seen from Eq. 4.274, the multiplication of the bulk temperature with the mass flow rate and the specific heat given the rate at which thermal energy is transported with the fluid as it moves along the tube.

With the above definition of the "bulk temperature", the local heat transfer coefficient is than defined as:

$$h = \frac{q''}{(t_w - t_b)} \qquad 4.275$$

where $t_w$ is the pipe wall temperature.

In practice, in a heated tube an energy balance may be used to determine the bulk temperature and its variation along the tube. Consider the tube flow seen in Fig. 4.22. The flow rate of the fluid and its inlet temperature (or enthalpy) to the tube are $\dot{m}$, respectively. Convection heat transfer occurs at the inner surface.

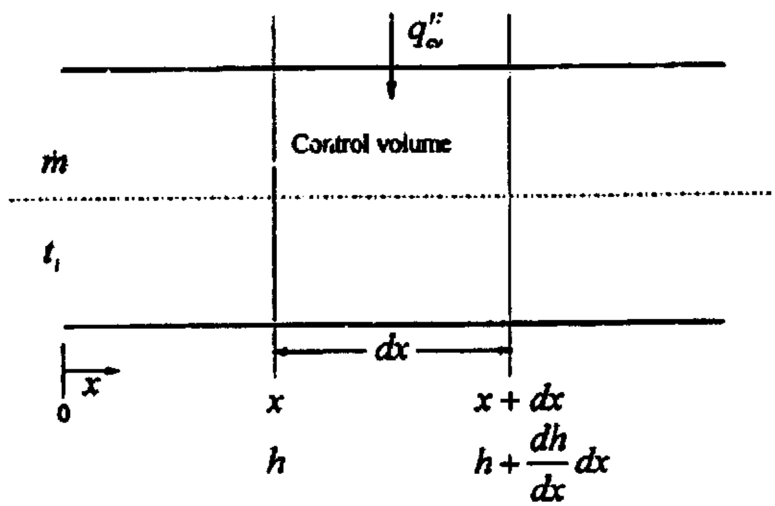



Figure 4.22 Control volumes for integral flow in a tube

Usually, fluid kinetic and potential energies, viscous dissipation and axial heat conduction are negligible. Steady state conditions prevail. Under these conditions the variation of the bulk temperature along the tube will be determined for "constant surface heat flux" and for "constant surface temperature".

1. Constant surface heat flux

Representing by $q''_w$ constant wall heat flux and applying the energy conservation equation (Eq. 4.64) to the control volume limited by the tube wall and by two planes perpendicular to the axis and a distance *dx* apart (Fig. 4.22), we can write:

$$q''_w \pi D dx = \dot{m}\left(h + \frac{dh}{dx}dx - h\right) \qquad 4.276$$

or

$$dh = \frac{\pi D}{\dot{m}} q''_w x \qquad 4.277$$

Integration of this equation between the inlet of the tube and a given axial position $(x)$ yields:

$$h(x) - h_i = \frac{\pi D}{\dot{m}} q''_w x \qquad 4.278$$

Enthalpy difference $h - h_i$ can be written in terms of temperature as:

$$h(x) - h_i = \bar{c}_p\left[t_b(x) - t_i\right] \qquad 4.279$$

Combining Eqs. 4.278 and 4.279, we obtain the variation of the bulk temperature along the tube as:

$$t_b(x) = t_i + \frac{\pi D}{\dot{m}} q''_w x \qquad 4.280$$

2. Constant surface temperature

Results for the axial distribution of the mean temperature are quite different when the temperature of wall is maintained at a constant value $t_w$. Under this condition the local wall heat flux is given by:

$$q''_w(x) = h_c\left[t_w - t_b(x)\right] \qquad 4.281$$

where $h_c$ is the convection heat transfer coefficient. Substituting Eq. 4.281 into Eq. 4.277 and knowing that:

$$dh = \bar{c}_p dt \qquad 4.282$$

we obtain:

$$\frac{dt}{t_w - t_b(x)} = t_i + \frac{\pi D}{\dot{m}} \frac{h_c}{C_p} dx \qquad 4.283$$

Integration of this equation yields:

$$\ln(t_w - t_b(x)) = \frac{\pi D}{\dot{m}C_p}\int_0^x h_c dx + C' \qquad 4.284$$

or

$$t_w - t_b(x) = \exp C' \cdot \exp\left(\frac{\pi D}{\dot{m}C_p}\int_0^x h_c dx\right) = C' \exp\left(\frac{\pi D}{\dot{m}C_p}\int_0^x h_c dx\right) \qquad 4.285$$

The constant $C$ can be determined using the boundary condition.

$$x = 0 \quad t_w - t_b = t_w - t_{bi} \qquad 4.286$$

as

$$C = t_w - t_{bi} \qquad 4.287$$

The variation of the bulk temperature along the tube is then given by:

$$\frac{t_w - t_b(x)}{t_w - t_{bi}} = \exp\left(\frac{\pi D}{\dot{m}C_p}\int_0^x h_c dx\right) \qquad 4.288$$

Defining the average value of $h_c$ from tube inlet to $x$ as

$$\bar{h}_c = \frac{1}{x}\int_0^x h_c dx \qquad 4.289$$

Eq. 4.288 is written as:

$$\frac{t_w - t_b(x)}{t_w - t_{bi}} = \exp\left(\frac{\pi D x}{\dot{m}C_p}\bar{h}_c\right) \qquad 4.290$$

This equation tells us that the temperature difference $t_w - t_b(x)$ decays exponentially with the distance along the tube axis. The temperature at the exit of the tube is obtained by setting $x = L$ in Eqs. 4.288 and 4.289 as:

$$\frac{t_w - t_b(x)}{t_w - t_{bi}} = \exp\left(\frac{\pi D L}{\dot{m}C_p}\bar{h}_{cL}\right) \qquad 4.291$$

where $\bar{h}_{cL}$ is the average value of $h_c$ for the entire tube.

Eq. 4.288 allows us to determine the bulk temperature $t_b(x)$ at a given axial position. If $h_c$ can be taken as constant along the tube, the determination of $t_b(x)$ is straightforward. If not, iterations are required to determine the value of $t_b(x)$ is straightforward. If not, iterations are required to determine the value of the bulk temperature. The bulk temperature concept introduced in this section is applicable to both laminar and turbulent flows in tubes.

#### 4.4.1.3 Heat Transfer in Fully Developed Laminar Flow

Although the analysis of the velocity distribution for fully laminar pipe flow is relatively simple, Eq. 4.262, the analysis of temperature distribution and consequently film coefficient is complex.

In a circular tube with uniform wall heat flux and fully developed laminar flow condition, it is analytically found that the Nusselt number is constant, independent of $\mathrm{Re}_D$, Pr and axial location: (Ozirik, 1985), i.e.,

$$Nu_D = \frac{h_c D}{k} = 4.364 \qquad 4.292$$

In this analytical derivation, it is assumed that the velocity distribution in the tube is given by Eq. 4.262 which is true for isothermal flows. For constant surface temperature conditions, it is also found that the Nusselt number is constant:

$$Nu_D = \frac{h_c D}{k} = 3.66 \qquad 4.293$$

Again, Eq. 4.262 is used for velocity distribution in this analytical analysis.

The use of a velocity distribution corresponding to isothermal flow condition is only valid for small temperature differences between the fluid and the tube wall. For large temperature differences, the fluid velocity distribution will be influenced by these differences as indicated in Fig. 4.23. Curve $b$ shows the fully developed parabolic distribution that would result for isothermal or very small temperature difference flow. When the heating is significant, the viscosity is lowest near the wall; as a result, the velocity increase (curve $a$)

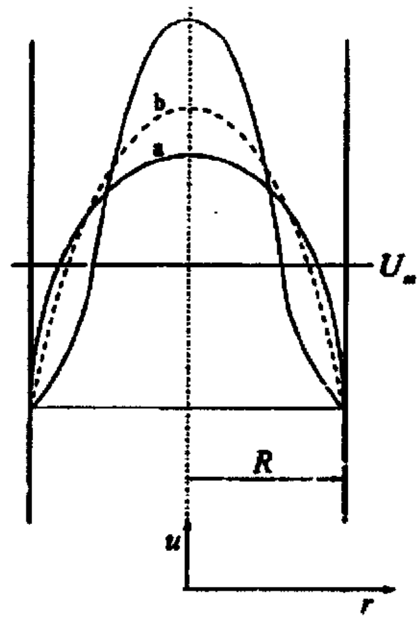



Figure 4.23 Influence of large temperature differences on velocity distributions in a tube

is faster than the isothermal case. (curve $b$). For the cooling, the reverse occurs as indicated by curve $c$ of Fig. 4.23: the viscosity is highest near the wall and the velocity increase is slower than the isothermal case. Consequently, although Eqs. 4.292 and 4.293 are enticing by their simplicity, for the above given reasons, they should not yield accurate convection heat transfer coefficient for laminar flows. Moreover, Eqs. 4.292 and 4.293 are only applicable to fully developed laminar flow conditions. As we discussed in laminar flows, the entrance region is quite substantial. In fact, in short tubes this region can occupy the entire length of the tube.

In parallel to the analytical studies, empirical relations have been developed to predict convection heat transfer for laminar flows in the entrance region of a circular tube. Hauseri(1943) developed the following empirical relation for laminar flow in a circular tube at constant wall temperature:

$$\overline{Nu_D} = \frac{\overline{h}_c D}{k_f} = 3.66 + \frac{0.0668(D/L)\mathrm{Re}_D \mathrm{Pr}}{1+0.04\left[(D/L)\mathrm{Re}_D \mathrm{Pr}\right]^{2/3}} \qquad 4.294$$

where $\overline{Nu_D}$ is the average Nusselt number. The above empirical relation approaches to the limiting value $\overline{Nu_D} = 3.66$ (Eq. 4.293) as the pipe length becomes very great compared with the diameter. In this relation the fluid properties are evaluated at the bulk temperature. Eq. 4.294 is recommended for:

$$\frac{\mathrm{Re}_D \mathrm{Pr}}{L/D} < 100$$

Sieder and Tale(1936) gave the following more convenient empirical correlation:

$$\overline{Nu_D} = 1.86\,\mathrm{Re}_D^{1/3}\mathrm{Pr}^{1/3}\left(\frac{D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \qquad 4.295$$

The fluid properties are to be evaluated at the bulk fluid temperature except for the quantity $\mu_w$, dynamic viscosity, which is evaluated at the wall temperature. The term $(\mu_b/\mu_w)^{0.14}$ is included to account for the fact that the boundary layer at the pipe surface is strongly influenced by the temperature dependence of the viscosity. This is particularly true for oils. The term $(\mu_b/\mu_w)^{0.14}$ applies to both heating and cooling cases. The effect of starting length is included in the term $(D/L)^{1/3}$. The range of applicability of Eq. 4.295:

$$0.48 < \mathrm{Pr} < 16{,}700$$

$$0.0044 < \frac{\mu_b}{\mu_w} < 9.75$$

$$\left(\frac{\mathrm{Re}_D \mathrm{Pr}}{L/D}\right)^{1/3}\left(\frac{\mu_b}{\mu_w}\right)^{0.14} \geq 2$$

### 4.4.2 Turbulent Flow

It is experimentally verified that one-seventh power law (Eq. 4.189), Blasius relation for shear stress on the wall (Eq. 4.192), $\delta_S / \delta$ ratio (Eq. 4.203) and $u_S / U$ ratio(Eq. 4.205) established for a turbulent flow in smooth tubes.

#### 4.4.2.1 Velocity Distribution and Friction Factor

The one-seventh power law given by Eq. 4.189, repeated here for convenience:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \qquad 4.189$$

can also be applied to turbulent flows in pipes by replacing

$$y \text{ by } R - r$$
$$\delta \text{ by } R \text{ or } D/2$$
$$U \text{ by } U_{max}$$

$U_{max}$ is the maximum velocity (Fig. 4.20). The velocity distribution for a pipe flow is then given by:

$$\frac{u}{U_{max}} = \left(\frac{R-r}{R}\right)^{1/7} \qquad 4.296$$

As in the case of the flat plate flow, this relation is approximate does not describe accurately the flow situation near the wall, it gives, however, a good representation of the gross behaviour of turbulent pipe flow. The average velocity is given by:

$$U_m = \frac{\int_0^R 2\pi r u dr}{\pi R^2}$$

$$= \frac{U_{max}}{\pi R^2}\int_0^R \left(\frac{R-r}{R}\right)^{1/7} dr = 0.817U_{max} \qquad 4.297$$

To obtain the pipe wall friction factor, we will use the Blasius relation given by Eq. 4.192 and repeated here for convenience:

$$\tau_w = 0.0228 \rho U^2 \left( \frac{\gamma}{U\delta} \right)^{1/4} \qquad 4.192$$

Replacing,

$\tau_w$ by $\tau_R$

$\delta$ by $D/2$

$U$ by $U_{max} = \frac{U_m}{0.817}$

we obtain:

$$\tau_R = 0.089 \rho U_m^2 \left( \frac{\nu}{U_m D} \right)^{1/4} \qquad 4.298$$

or using the definition of the friction factor given by Eq. 4.267

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2} = \frac{0.312}{\left( \frac{U_m D}{\nu} \right)^{1/4}} = \frac{0.312}{\mathrm{Re}_D^{1/4}} \qquad 4.299$$

The above relation fits the experimental data well for:

$$10^4 < \mathrm{Re}_D < 5 \times 10^4$$

If the constant 0.312 is replaced by 0.316, the upper limit can be extended to $10^5$. For higher $\mathrm{Re}_D$ numbers the following relation can be used:

Prandtl Equation

$$\frac{1}{\sqrt{f}} = 2.0 \log \left( \mathrm{Re}\sqrt{f} \right) - 0.8 \qquad 3000 < \mathrm{Re}_D < 3.4 \times 10^6 \qquad 4.300$$

von Karman Equation

$$\frac{1}{\sqrt{f}} = 2.0 \log \left( D/E \right) + 1.74 \qquad \frac{D}{\varepsilon} \frac{1}{\mathrm{Re}\sqrt{f}} > 0.01 \qquad 4.301$$

Several friction factor correlations are available in the literature.

For turbulent flows, the frictional pressure gradient is given by the came expression obtained for laminar flows (Eq. 4.268) repeated here:

$$-\frac{dp}{dx} = f \frac{1}{2} \rho U_m^2 \frac{1}{D} \qquad 4.268$$

The only difference is that the friction factor, $f$, will be determined by using one of the correlations given above (Eqs. 4.299-4.301) or any other ad-hoc friction factor correlation available in the literature

#### 4.4.2.2 Heat Transfer in Fully developed Turbulent Flow

The heat transfer correlation established in Section 4.3.2.2 for a flat plate (Eq. 4.237), repeated here for convenience, is given by:

$$h = \frac{\dfrac{\tau_w C_p}{U}}{1 + \dfrac{u_S}{U}(\Pr - 1)} \qquad 4.237$$

This equation was based on Reynolds' analogy as modified by Prandtl. It was assumed that the turbulent boundary layer consists of two layers: a viscous layer where the molecular diffusivity is dominant. This structure should remain the same for turbulent flows in pipes. In order to apply Eq. 4.237 to turbulent flow in pipes, the following adjustments will be made:

$U$ will be replaced by $U_m$ (Eq. 4.297), and

$\tau_w$ will be replaced by $\tau_R$

The velocity ratio $u_S / U$ given by Eq. 4.205

$$\frac{u_S}{U} = 1.878\left(\frac{\rho U \delta}{\mu}\right)^{-1/8} \qquad 4.205$$

will be adapted to pipe flow condition by interpreting:

$U$ as $U_{max}$, $U_{max} = \dfrac{U_m}{0.817}$ (Eq. 4.297), and

$\delta$ as $D/2$

Under these conditions, $u_S / U$ ratio given by Eq. 4.205 becomes:

$$\frac{u_S}{U} = 2.44\left(\frac{\mu}{\rho U_m D}\right)^{1/8} = \frac{2.44}{\mathrm{Re}_D^{1/8}} \qquad 4.302$$

Using the definition of the friction factor (Eq. 4.267)

$$f = \frac{4\tau_R}{\frac{1}{2}\rho U_m^2} \qquad 4.267$$

the wall shear stress $\tau_R$ can be written as:

$$\tau_R = \frac{f}{8}\rho U_m^2 \qquad 4.303$$

We will assume that $f$ is given by:

$$f = \frac{0.316}{\mathrm{Re}^{0.25}} \qquad 4.304$$

Replacing in Eq. 4.287, $U$ by $U_m$, using Eqs. 4.302, 4.303 and 4.304, and introducing the Nusselt number ( $Nu = h_c D / k_f$ ) and diameter Reynolds number ($\mathrm{Re}_D = \rho U_m D / \mu$), we obtain:

$$Nu_D = \frac{0.0396\ \mathrm{Re}_D^{3/4}\,\mathrm{Pr}}{1 + 2.44\,\mathrm{Re}_D^{-1/8}(\mathrm{Pr} - 1)} \qquad 4.305$$

Experiments show that the relation works reasonably well in spite of the simplifying assumptions that have been made in its derivation. It is usually suggested (Hoffman, 1938) that the constant 2.44 in Eq 4.305 be replaced by:

$$1.5\mathrm{Pr}^{-1/6} \qquad 4.306$$

Consequently, Eq. 4.305 becomes:

$$Nu_D = \frac{0.0396\ \mathrm{Re}_D^{3/4}\,\mathrm{Pr}}{1 + 1.5\,\mathrm{Pr}^{-1.6}\,\mathrm{Re}_D^{-1/8}(\mathrm{Pr} - 1)} \qquad 4.307$$

Both of these relations are based on the determination of the fluid properties at the bulk fluid temperature. Eqs. 4.305 and 4.307 give good results for fluids with $\mathrm{Pr}$ - number close to 1.

If the difference between the pipe surface temperature and the bulk fluid temperature is smaller than 6C for liquids and 60C for gases, the following empirical correlation based on the bulk temperature can be used:

$$Nu_D = 0.023\ \mathrm{Re}_D^{0.8}\,\mathrm{Pr}^n \qquad 4.308$$

where

$$n = 0.4 \text{ for heating}$$
$$n = 0.3 \text{ for cooling}$$

This correlation is applicable for smooth pipes for

$$0.7 < \Pr < 160$$
$$\text{Re} > 10{,}000$$
$$\frac{L}{D} > 60$$

For temperature difference greater than those specified above or for fluids more viscous than water, Sieder and Tale (1936) proposed:

$$Nu_D = 0.027 \, \text{Re}_D^{0.8} \Pr^{1/3} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \tag{4.309}$$

All properties are evaluated at the bulk fluid temperature except for $\mu_w$ which is to be evaluated at the pipe wall temperature. Range of applicability:

$$0.7 < \Pr < 16{,}700$$
$$\text{Re} > 10{,}000$$
$$\frac{L}{D} > 60$$

The previous heat transfer correlations give a maximum errors of ± 25% in the range of $0.67 < \Pr < 100$ and apply to turbulent flow in smooth tubes. An accurate correlation applicable to rough ducts has been proposed by Petukhow:

$$Nu_D \frac{\text{Re}_D \Pr}{X} \left( \frac{f}{8} \right) \left( \frac{\mu_b}{\mu_w} \right)^n \tag{4.310}$$

$$X = 1.07 + 12.7 \left( \Pr^{2/3} - 1 \right) \left( \frac{f}{8} \right)^{1/2} \tag{4.311}$$

for liquids:

$$n = 0.11 \text{ for heating}$$
$$n = 0.25 \text{ for cooling}$$

for gases: $n = 0$. The range of applicability of the Petukhow correlation is:

$$10^4 < \mathrm{Re}_D < 5 \times 10^6$$
$$2 < \mathrm{Pr} < 140 \quad \approx 5\% \text{ error}$$
$$0.5 < \mathrm{Pr} < 2000 \quad \approx 10\% \text{ error}$$
$$0.08 < \frac{\mu_w}{\mu_b} < 40$$

All physical properties, except $\mu_w$, are evaluated at the bulk temperature. $\mu_w$ is evaluated at the wall temperature. The friction factor $f$ can be determined using an adequate correlation such as Eq. 4.300 or 4.301.

## 4.5 Non-Circular Tubes

So far, discussion of the friction factor, frictional pressure gradient and heat transfer coefficient for turbulent flow has been limited to flow in circular tubes. However, in engineering application the flow section is non-circular. The correlations for friction factor as well as for heat transfer coefficient presented above may be applied to non-circular tubes if the diameter appearing these correlations is replaced by the hydraulic diameter of the non-circular duct defined as:

$$D_h = \frac{4A}{P} \qquad 4.312$$

where $A$ is the cross-sectional area for the flow and $P$ is the wetted parameter.

For example, the hydraulic diameter of an annular cross section with inner diameter $D_1$, and outer diameter $D_2$ is given by:

$$D_h = \frac{4 \cdot \frac{\pi}{4}\left(D_2^2 - D_1^2\right)}{\pi\left(D_2 + D_1\right)} = D_2 - D_1 \qquad 4.313$$