Mathematics - Course 221

THE DERIVATIVE IN SCIENCE AND TECHNOLOGY

I Some Common Differential Equations

Calculus is to modern science and technology as arithmetic is to accounting. Arithmetic provides the notation and techniques for computing credits, debits and balance on hand; calculus provides the notation and techniques for computing the 'instantaneous' or true rate of change of one physical variable with respect to another, given the mathematical relationship between the two variables. Without calculus, only average rates of change can be calculated, in general. (Calculus provides also the notation and techniques for solving the inverse problem of finding a function, given its rate of change, as will be seen in the next two lessons.)

Differential calculus has been applied previously in this text to the following topics: velocity, acceleration, nuclear decay, and reactor power growth. To illustrate the general applicability of calculus, a few of the most common *differential equations* from the fields of mechanics, electricity, nuclear theory, heat and thermodynamics, and magnetism are listed in Table 1. (A differential equation is simply an equation involving at least one derivative.) Trainees will have seen many of these rate-of-change statements in non calculus form in previous courses.

Note that "t" represents time throughout Table 1.

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TABLE 1

Some Common Differential Equations of Science

Differential Equation	Definition of Variables
$F = m \frac{dv}{dt}$ (Newton's Second Law)	F is force m is mass v is velocity
$\omega = \frac{d\theta}{dt}$	ω is angular velocity θ is angular displacement
$\alpha = \frac{d\omega}{dt}$	α is angular acceleration ω is angular velocity
$\tau = I\frac{d\omega}{dt} = I\alpha$	τ is torque I is moment of inertia ω, α as above
$F = -\frac{dE_p}{dr}$	F is force E_p is potential energy in a central force field as a function of r, the distance from the center of force (eg, gravity, Coulomb electric forces)
$P = \frac{dW}{dt}$	P is power W is energy converted or work done
$i = \frac{dq}{dt}$	i is electric current q is charge
$i_c = C \frac{dV_c}{dt}$	i_c is capacitor current flow C is capacitance of capacitor V_c is capacitor voltage

Differential Equation	Definition of Variables
$V_L = L \frac{di_L}{dt}$	V_L is inductor voltage L is inductance i_L is inductor current
$\frac{dN}{dt} = -\lambda N$	N is number of radioactive nuclei remaining at time t λ is the decay constant
$\frac{dA}{dt} = -\lambda A$	A is radioactive source activity λ as above
$\frac{dN}{dx} = -\Sigma x$	Attenuation of nuclear radiation: N is number nuclear projectiles (neutrons, γ's, β's, etc) having penetrated to depth x Σ is macroscopic cross section of attenuating material
$\frac{dP}{dt} = \frac{\Delta k}{L} P$ $\frac{d}{dt} \ln P = \frac{\Delta k}{L}$	P is reactor power Δk is reactivity L is mean neutron lifetime
$C = \frac{1}{m} \frac{dQ}{dT}$	C is specific heat capacity of a substance m is mass of substance Q is quantity of heat stored in substance (in joules) T is temperature of substance ($^\circ$A)
$\frac{dQ}{dt} = C\Delta T \frac{dm}{dt}$	ΔT is temperature difference (assumed constant in this equation) Q, m, C as above

Differential Equation	Definition of Variables
$H = -kA \frac{dT}{dx}$	H is the heat flow rate through a medium ($\frac{J}{S}$) k is the thermal conductivity of the medium A is the cross sectional area of the medium T is the temperature (^{0}A) x is the penetration depth into the conducting medium
$\frac{dV}{dT} = \frac{nR}{P}$ (Ideal Gas Law)	P is pressure T is temperature (^{0}A) n is number of moles R is gas constant
$V = -N \frac{d\Phi}{dt}$ (Faraday's Law)	V is the voltage across a coil N is number turns in the coil Φ is the magnetic flux through the coil

II Some Instruments Which Differentiate

1) A vibrometer gives the rms velocity of a vibration. This is the average value of the square root of the square of the derivative of the displacement of the pick-up attached to a vibrating object. A second output from the vibrometer gives the acceleration (derivative of the derivative of the displacement) of the vibration.

2) A magnetic phono cartridge delivers a voltage whose amplitude is proportioanl to the time derivative of the displacement of the stylus.

3) An accelerometer is a transducer that provides an output voltage proportional to the acceleration of some object. This voltage is produced across a piezoelectric crystal.

4) An operational amplifier in the differentiating mode produces an output voltage proportional to the derivative of its input (see Figure 1).

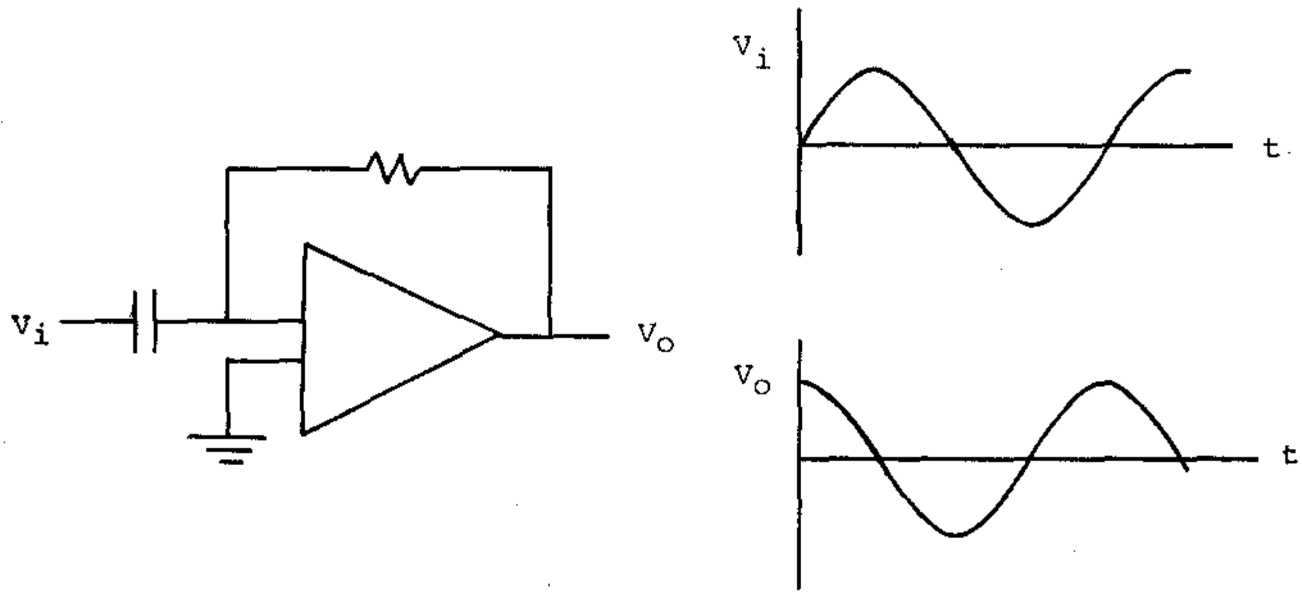


Figure 1

III Derivative Control Mode

Negative feedback is used to control process parameters such as pressure, flow, temperature, reactor power, etc., to set point. The difference between the *set point* and actual *measured value* is called the *error*. For example, suppose the fluid level in the tank of Figure 2 is to be controlled at the set point. The flow demand from the bottom of the tank is variable, and the level is maintained at the set point by manipulating the inflow rate via the inlet control valve. The level measurement is supplied to the controller by the level transmitter (LT). The controller establishes the difference between this measured value and the set point, ie, the error, and sends a control signal to the control valve actuator. This control signal always manipulates the inflow to the tank in such a way as to reduce the error, ie, if the level is below set point, the controller opens the inlet valve, and vice versa.

If the control signal (CS) is directly proportional to the error, e, control is said to be *proportional*, ie,

$(CS)_p = k_p e \ (+\ b)$, where k_p is a constant and b is the constant equilibrium bias

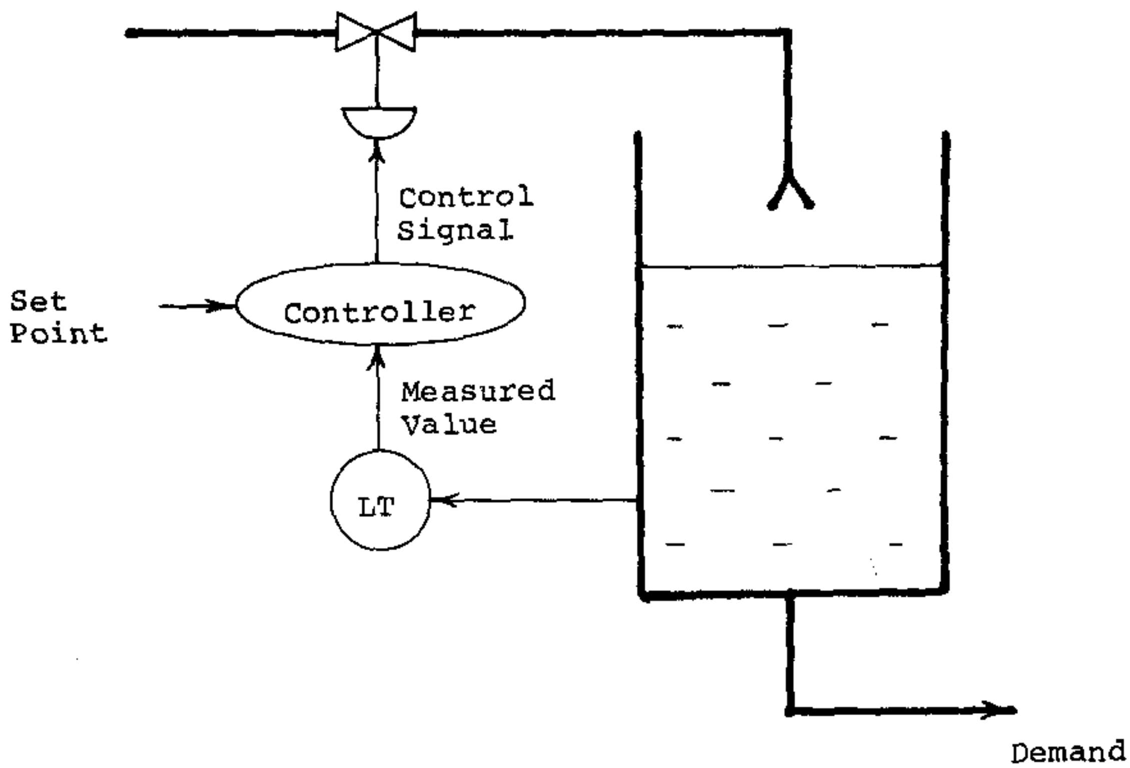


Figure 2 Tank Level Control Loop

The disadvantages of proportional control are:

(1) The controller cannot begin to take corrective action until after an error is established. Because of time lag in the control loop, this often leads to considerable 'overshoot' in corrective action (see Figure 4), with process instability while the controller 'hunts' for the correct control signal to match inlet and outlet flows.

(2) No control signal (other than equilibrium bias) is possible without an error. For example, if demand increases, the level must fall sufficiently below set point that the control signal, $(CS)_P$, can match inflow and outflow. The amount by which the level must remain below set point in order to keep the flows matched (see Figure 4) is called the *offset*.

The first of the above disadvantages can be counteracted with the use of the *derivative control mode*; the second disadvantage is overcome with the use of the *reset control mode* (see 221.30-2, section VIII). A derivative controller's output is proportional to the rate at which the level is straying from set point, ie, to rate of change of the error:

$$(CS)_D = k_D \frac{de}{dt}$$

Graphically speaking, the derivative mode control signal is proportional to the slope of the tangent to the error-time curve. In practice, derivative mode control is usually used in conjunction with proportional mode, so that the control signal is made up of proportional and derivative components:

$$(CS)_{PD} = k_P e + k_D \frac{de}{dt} \quad (+\ b)$$

The rate signal, $\frac{de}{dt}$, can be obtained by passing the error signal e(t) through a differentiating amplifier.

The proportional and derivative components and the total output signal from a proportional-derivative controller are shown in Figure 3 for two hypothetical examples of level fluctuations in the tank of Figure 2. (NB: assume that demand varies such that the level fluctuates as shown in spite of feedback). In Figure 3(a), the tank level drops linearly from set point to a lower value, and in Figure 3(b), the level makes a temporary excursion below set point. Note that a few representative tangents have been drawn on the error curve of Figure 3(b), and that the derivative component of the control signal correlates with the value of the tangent slope at every instant in both Figures 3(a) and 3(b).

Note that in both Figures 3(a) and 3(b), as soon as the level <u>starts</u> to drop, the controller immediately puts out a signal <u>via</u> the derivative mode component to open the inlet valve. The amplitude of this signal is proportional to the <u>rate</u> at which the level is dropping. This is in contrast to the proportional mode component, which becomes significant only after the error grows significantly. Thus derivative control builds an 'anticipatory' feature into the control loop. In fact, an output proportional to the <u>present rate</u> of error growth is actually the same output that <u>would</u> be <u>produced</u> by the proportional mode at some later time (how <u>much</u> later depends on the constant, k_D), assuming that the error <u>were</u> to continue to grow at the same rate in the meantime. Graphically speaking, the derivative mode controller extrapolates a certain number of seconds along the tangent to the error-time curve.

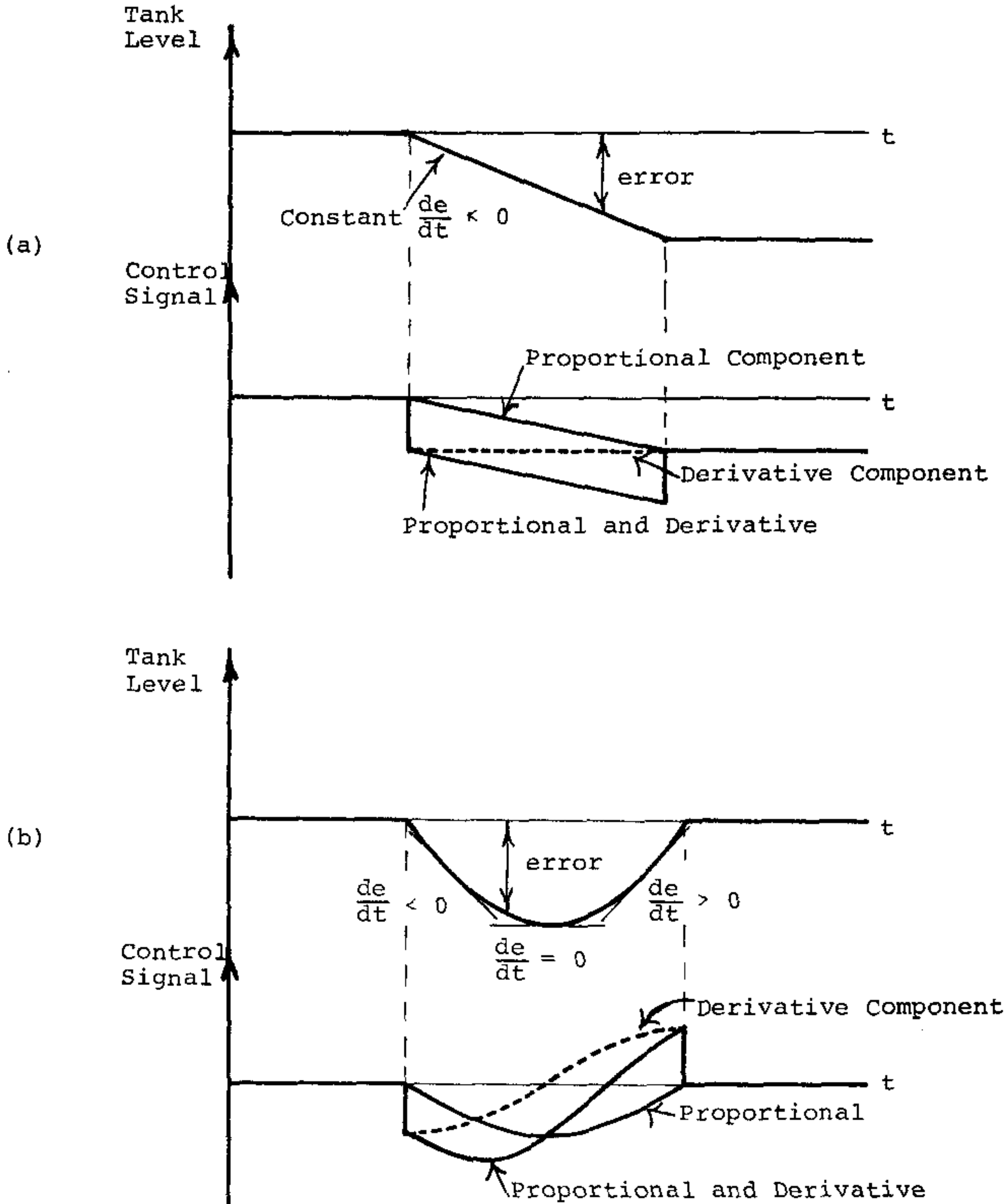


Figure 3 Proportional and Derivative Control Signals for Hypothetical Tank Level Fluctuations

Figure 4 shows typical fluctuations in level following a step increase in demand

(a) without any feedback control - the level drops at a constant rate

(b) with proportional control - the level drops to offset value, overshoots, oscillates, and eventually settles out at offset value

(c) with proportional plus derivative control - the level stabilizes to the same offset value more rapidly, with reduced overshoot.

To summarize, the advantage of derivative mode control is that corrective action is initiated before a large error is established, and faster stabilization is achieved with smaller deviation from set point following process transients. Arguments analogous to the preceeding discussion of tank level control indicate similar advantages to using derivative control mode in controlling reactor power, boiler level, moderator temperature, etc., in CANDU plants.

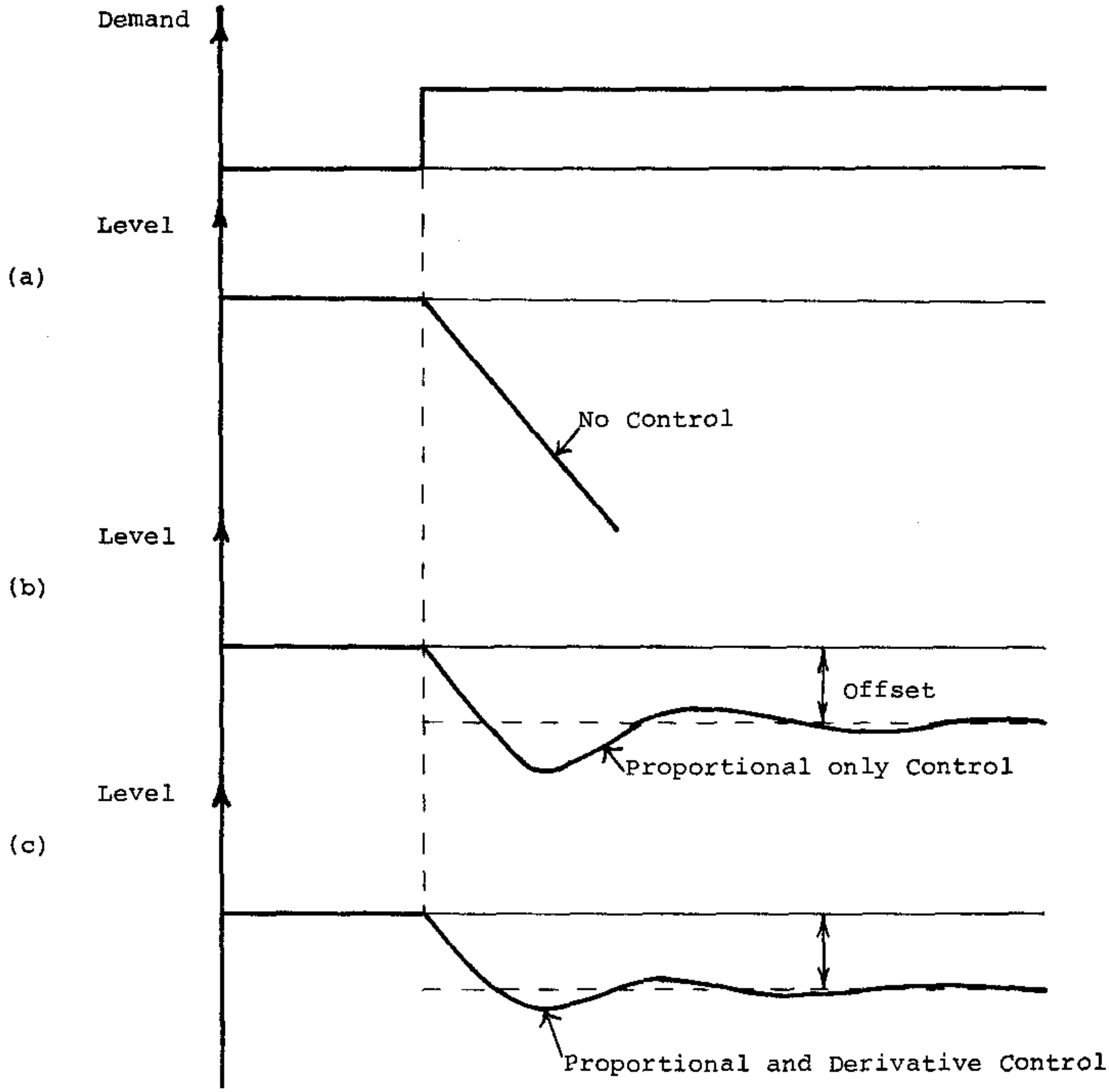


Figure 4 Tank Level Fluctuations Following a Step Demand Increase

ASSIGNMENT

1. Write a verbal rate-of-change statement corresponding to each differential equation in Table 1 of this lesson.

2. Practice writing the differential equations corresponding to the rate-of-change statements given in section 221.40-4 in answer to question #1 above.

3. A solenoid moves a plunger s meters in t seconds according to the relation

$$s(t) = 2t^3 + 0.02t$$

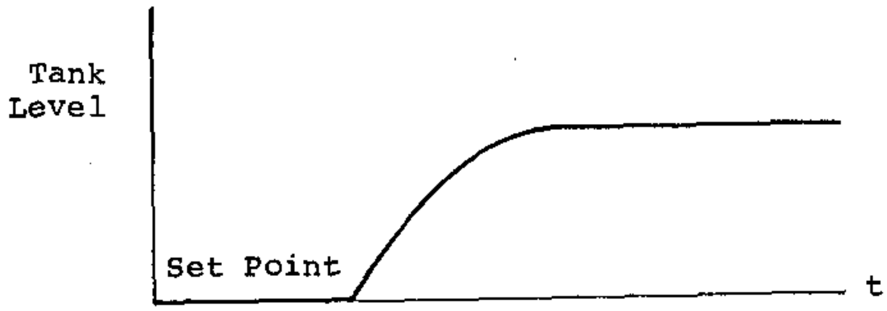
   If the plunger mass is 0.05 kg, calculate the force exerted on the plunger at $t = 0.01s$.

4. (a) Write the differential equation corresponding to the following rate-of-change statement: "The voltage V_2 induced in coil #2 equals the product of the mutual inductance M between coils #1 and #2 times the rate of decrease of the current i_1 through coil #1".

   (b) If M = 2 henries and $I_1 = 3t^2 - t^3$ amperes, calculate (i) $V_2(t)$, (ii) $V_2(2)$.

5. The following diagram depicts a hypothetical variation in the tank level of Figure 2. On the same time axis, sketch the following for a proportional-derivative controller:

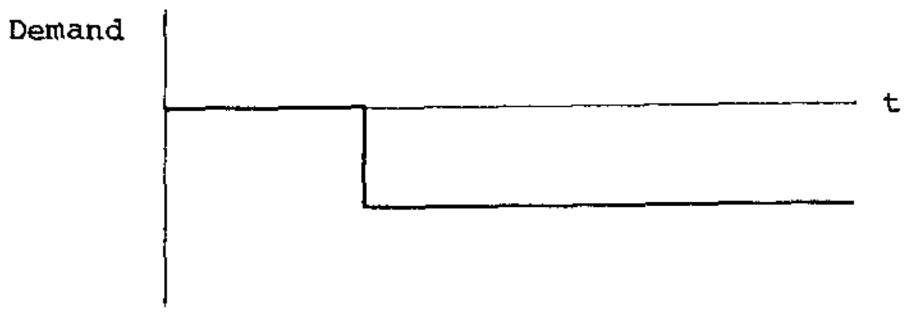
   (a) proportional component of control signal
   (b) derivative component of control signal
   (c) total output signal.



6. The following diagram depicts a step decrease in the demand flow from the tank of Figure 2. On the same time axis, sketch typical corresponding fluctuations in tank level in the following cases:

   (a) no level control
   (b) proportional only level control
   (c) proportional-derivative level control

   Label the offset in (b) and (c). Assume level was at set point prior to demand change.



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