4 - Elementary Solutions to the Kinetics Equations

- Point Kinetics Equations

  - amplitude $p(t)$

$$\frac{dp}{dt}(t) = \left[\frac{\rho(t)-\beta}{\Lambda}\right]p(t) + \frac{1}{\Lambda}s_d(t) + \frac{1}{\Lambda}s(t)$$

  - precursors $\zeta(t)$

$$\frac{d\zeta_k}{dt}(t) = -\lambda_k \zeta_k(t) + \beta_k p(t)$$

  - delayed neutron source: $s_d(t)$

$$s_d(t) = \sum_k \lambda_k \zeta_k(t)$$

- Within the limitations of Point Kinetics, these equations are exact and the reactivity can vary arbitrarily. We will first seek *analytical solutions* to these equations, starting with the most simple representation, a step-change in reactivity (i.e. *constant* reactivity)

- Steady-State:

  Any transient starts with an initial steady-state. Setting, dp/dt and dζ/dt equal to 0:

$$(\rho_o - \beta)p_o + \sum_k \lambda_k \zeta_{ko} + s_o = 0$$

$$-\lambda_k \zeta_{ko} + \beta_k p_o = 0$$

$$s_{do} = \sum_k \lambda_k \zeta_{ko} = \beta p_o$$

  Note: if reactor is critical, ρ_o=0 and s_o=0

Initial Steady State

Source Multiplication Formulas:

- **subcritical**

$$p_o = \left(\frac{1}{-\rho_o}\right) s_o$$

The factor $(1/-\rho_o)$ is the *subcritical multiplication factor.* Note that s_o is independant of ρ_o. The flux level in a subcritical reactor is not arbitrary, but is proportionnal to the intensity of the *external* source.

- **critical**

$$p_o = \left(\frac{1}{\beta}\right) s_{do}$$

The factor $(1/\beta)$ acts as a multiplication factor for the delayed source in a critical reactor. Note that s_{do} that is not independent of β. We can interpret this equation by saying that the delayed source in a critical reactor *at steady state* is proportionnal to the flux level.

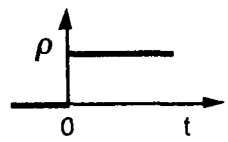
- **generally :**

$$p_o = \frac{s_o + s_{do}}{\beta - \rho_o}$$

The factor $1/(\beta - \rho_o)$ is the *source multiplication factor*, which includes the delayed source. Again we can say that the flux level in a reactor *at steady state* is proportionnal to the total source, external + delayed.

Constant Reactivity (step-change) with one delayed group

- Initially critical reactor



$$\rho(t) = \begin{cases} 0 & (t < 0) \\ \rho_o & (t \geq 0) \end{cases}$$

- take the Laplace transform of the P.K. equations, with one delayed group (K=1)

$$sP(s) - p(0) = a\,P(s) + \frac{\lambda}{\Lambda}X(s)$$
$$s\,X(s) - \zeta(0) = \beta\,P(s) - \lambda\,X(s)$$

where

$$a = \frac{\rho_o - \beta}{\Lambda}$$

and

$$P(s) = \mathcal{L}\{p(t)\}$$
$$X(s) = \mathcal{L}\{\zeta(t)\}$$

- We find:

$$X(s) = \frac{\beta}{s+\lambda}\left[P(s) + \frac{p_o}{\lambda}\right]$$

$$P(s) = p_o\left[\frac{1 + \dfrac{\beta}{\Lambda(s+\lambda)}}{s - a - \dfrac{\lambda\beta}{\Lambda(s+\lambda)}}\right]$$

$$= p_o\left[\frac{s + \lambda + \beta/\Lambda}{(s-a)(s+\lambda) - \lambda\beta/\Lambda}\right]$$

$$= p_o\left[\frac{s + \lambda + \beta/\Lambda}{s^2 + (\lambda - a)s - \lambda\underbrace{(a + \beta/\Lambda)}_{\rho_o/\Lambda}}\right]$$

$$= p_o\left[\frac{s + \lambda + \beta/\Lambda}{(s-\omega_1)(s-\omega_2)}\right]$$

Constant Reactivity (cont'd)

- The poles of $P(s)$ are the roots ω_1 and ω_2 of the characteristic polynomial:

$$s^2 + (\lambda - a)s - \frac{\lambda \rho_o}{\Lambda} = 0$$

that is:

$$\omega_1 = \frac{(a-\lambda)}{2} + \sqrt{\frac{(\lambda - a)^2}{4} + \frac{\lambda \rho_o}{\Lambda}}$$

$$\omega_2 = \frac{(a-\lambda)}{2} - \sqrt{\frac{(\lambda - a)^2}{4} + \frac{\lambda \rho_o}{\Lambda}}$$

- The solution for constant reactivity in the frequency domain can then be written:

$$P(s) = p_o\left(\frac{A_1}{s-\omega_1} + \frac{A_2}{s-\omega_2}\right)$$

where:

$$A_1 = \lim_{s \to \omega_1} \left\{(s-\omega_1)\frac{P(s)}{p_o}\right\}$$

$$A_2 = \lim_{s \to \omega_2} \left\{(s-\omega_2)\frac{P(s)}{p_o}\right\}$$

Constant Reactivity (cont'd)

- In the time domain, the general solution will be:

$$p(t) = p_o\left[A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t}\right]$$

with

$$A_1 = \frac{1}{\omega_1 - \omega_2}\left(\frac{\rho_o}{\Lambda} - \omega_2\right)$$

$$A_2 = \frac{1}{\omega_1 - \omega_2}\left(\omega_1 - \frac{\rho_o}{\Lambda}\right)$$

- the (total) precursor concentration will then be:

$$\zeta(t) = \beta p_o\left\{\frac{e^{-\lambda t}}{\lambda} + A_1\left(\frac{e^{\omega_1 t} - e^{-\lambda t}}{\omega_1 + \lambda}\right) + A_2\left(\frac{e^{\omega_2 t} - e^{-\lambda t}}{\omega_2 + \lambda}\right)\right\}$$

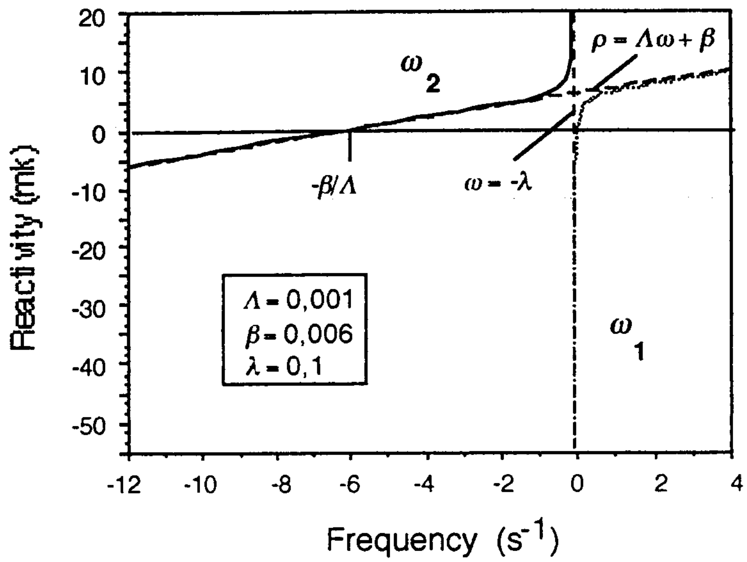
Typical values of the one-group parameters

	CANDU	PWR	FBR
Λ (s)	0,001	< 0,0001	< 0,00005
β	0,006	0,0075	0,0035
λ (s^{-1})	0,1	0,1	0,1

Analytical expression for a reactivity echelon

$$\rho(\omega) = \left[\Lambda + \frac{\beta}{\omega + \lambda}\right]\omega = \rho_o$$

- Graphically:



- We observe that the two roots are always widely seperated:

$$\left|\frac{\omega_1}{\omega_2}\right| >> 1 \quad \text{si } \rho_o > \beta$$

$$\left|\frac{\omega_2}{\omega_1}\right| >> 1 \quad \text{si } \rho_o < \beta$$

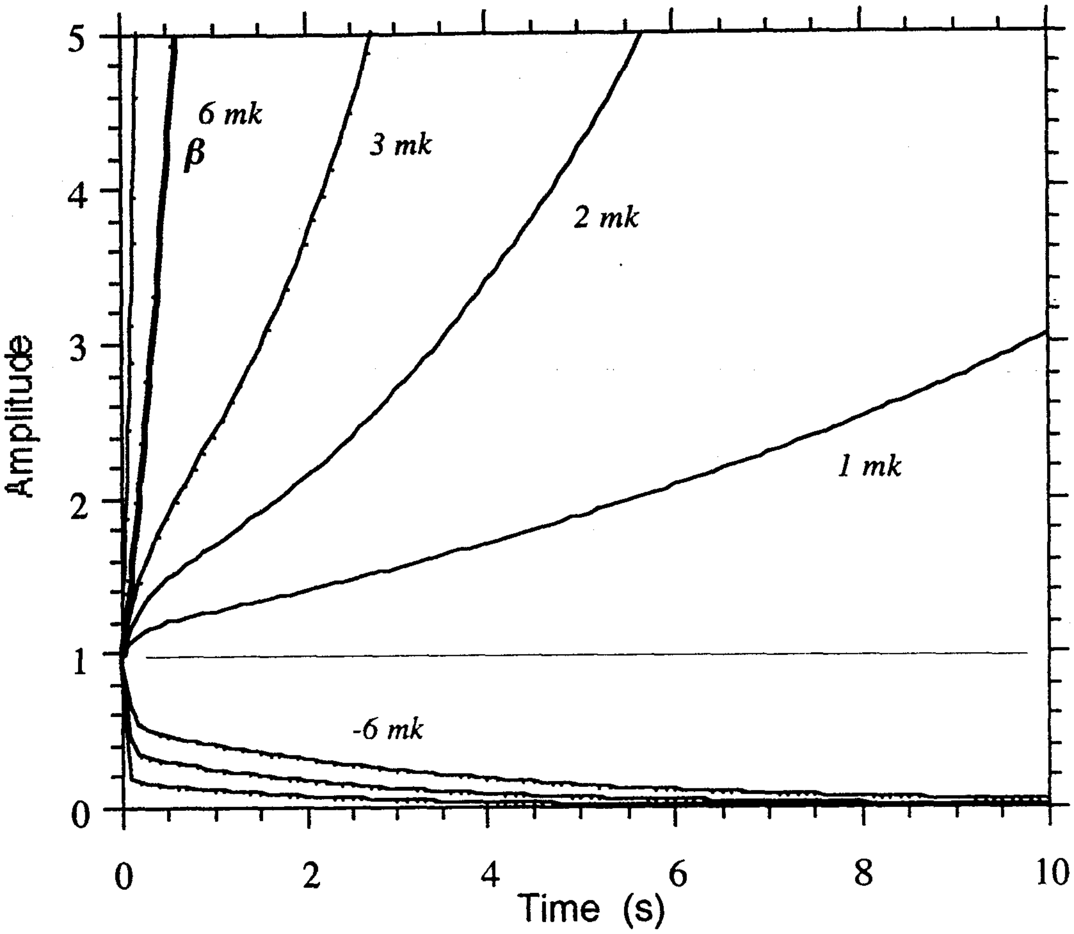
- This implies that one of the two exponentials will rapidely dominate the solution.

- An approximate analytical solution, valid within 1% as long as ρ is not too close to β:

$$p(t) \approx \rho_o \left[\left(\frac{\beta}{\beta - \rho_o}\right) e^{\left(\frac{\lambda\rho_o}{\beta-\rho_o}\right)t} - \left(\frac{\rho_o}{\beta - \rho_o}\right) e^{\left(\frac{\rho_o-\beta}{\Lambda}\right)t} \right]$$

as long as $|\rho_o - \beta| \geq \frac{\beta}{2}$

Response to a positive echelon in CANDU



- As expected, power increases when $\rho>0$ (supercritical), and it decreases when $\rho<0$ (subcritical);

- We note a completely different response of the reactor when $\rho \geq \beta$:

  - When $\rho > \beta$, the reactor is prompt supercritical. The reactor would be supercritical even without the delayed source. The prompt multiplication of neutrons dominates the response.

  - The rate of increase of power is extremely rapid. The solution is dominated by the second exponential, which becomes positive when $\rho > \beta$.

  - When $\rho < \beta$, the reactor is prompt subcritical. Power increases or decreases depending on the sign of

Prompt Subcritical State ($\rho > \beta$)

- The response is characterized by a prompt jump (or drop), followed by a gradual increase (or decrease).

- The prompt jump is caused the rapid adjustment of the (subcritical) prompt multiplication. The second exponential vanishes rapidly.

- During the first fraction of a second, the delayed neutron source s_d has not varied significantly. The amplitude of the prompt jump can then be estimated by postulating that s_d is a constant in the amplitude equation:

$$\frac{dp}{dt} = \frac{(\rho_o - \beta)}{\Lambda} p + \frac{s_{do}}{\Lambda} \Rightarrow 0$$

$$\therefore \; p^+ \Rightarrow \frac{s_{do}}{\beta - \rho_o} = \frac{\beta}{\beta - \rho_o} p_o$$

- After the prompt jump, neutron power in the reactor essentially responds to the evolution of the delayed source (precursors). Since the concentrations of precursors are governed by relatively large decay time constants, power varies much more slowly.

Reactor Period vs. Reactivity

- Definition of ***reactor period***,

$$\frac{1}{T} = \frac{1}{p(t)}\frac{dp}{dt}$$

- After the initial transient, there is a fixed relation between period and reactivity:

  - if $\rho<\beta$:

$$\boxed{T = \frac{\beta-\rho}{\lambda\rho}}$$

  - if $\rho>\beta$:

$$\boxed{T = \frac{\Lambda}{\rho-\beta}}$$

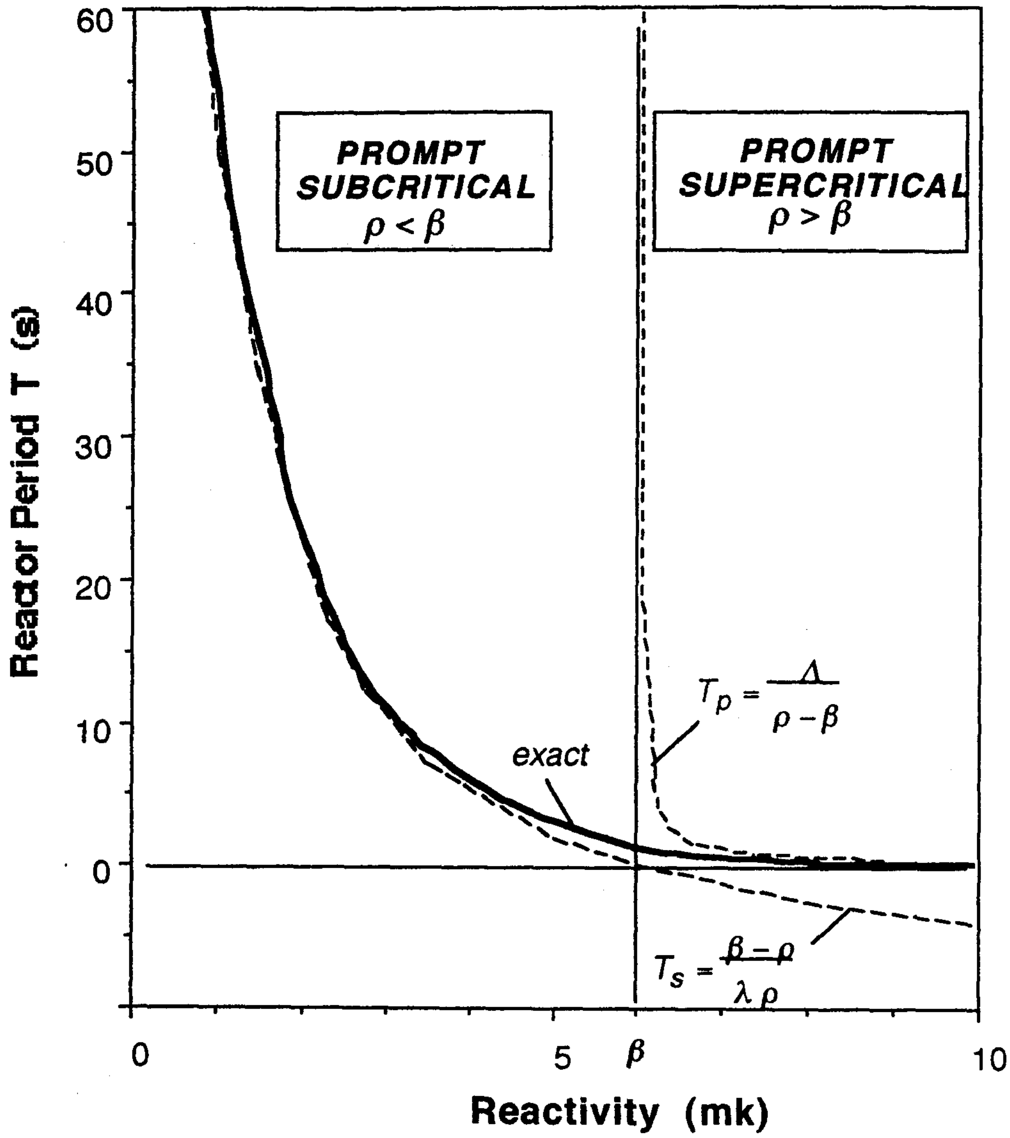
- We observe:

  - when $\rho < 1$ mk, the period is *larger than 1 minute*
  - when $\rho \approx \beta$ (6 mk), the period is of the order of $\approx$ *1 second.*
  - This behavior is common to all reactor systems

- It implies that:

  - **there is no problem in controlling nuclear reactors when reactivity is maintained well below $\rho = \beta$;**
  - **prompt-criticality must be avoided at all costs (Chernobyl)**

Reactor Period vs. Reactivity

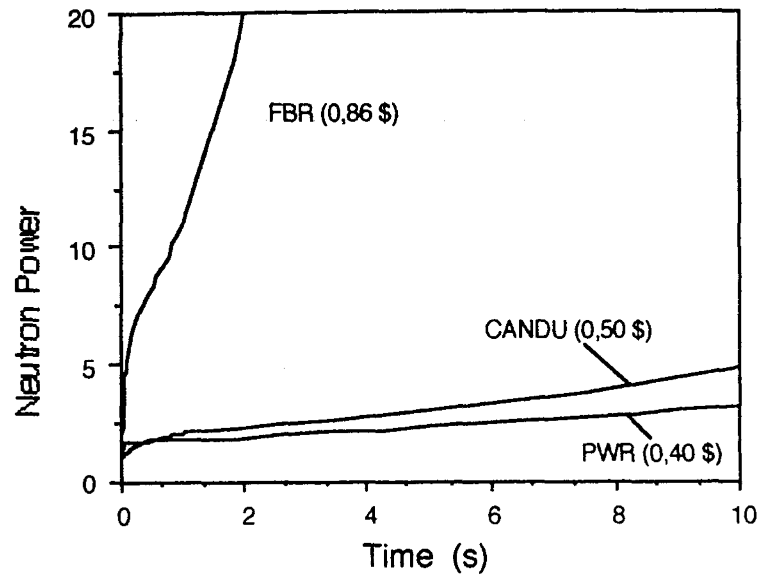


Comparison with other reactor types

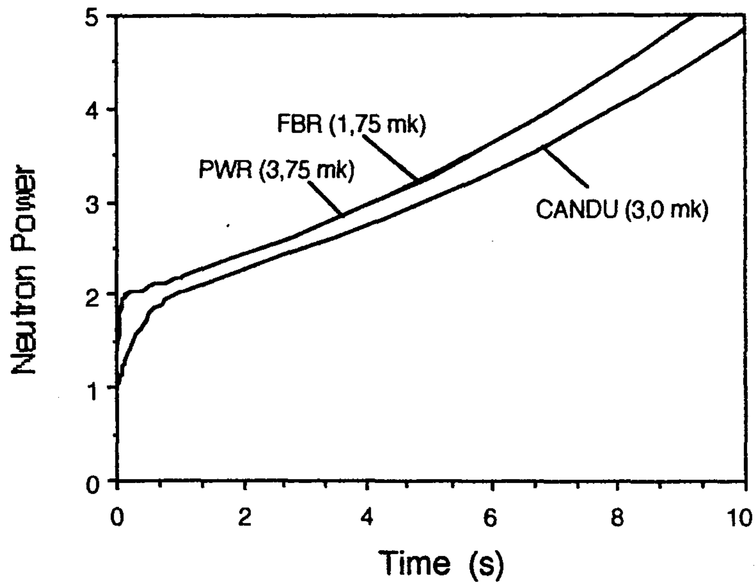
- reactivity can be expressed in $, by simply considering :

$$\rho(\$) = \frac{\rho}{\beta}$$

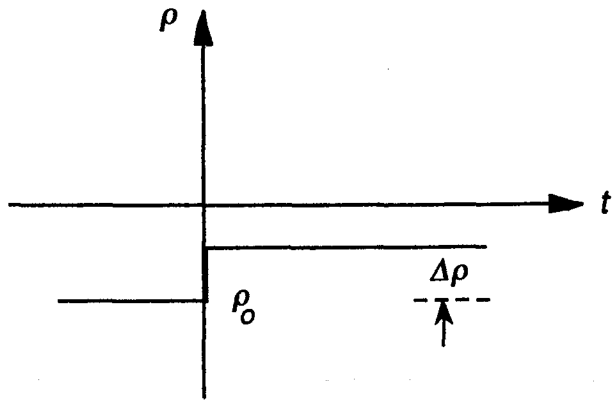
a) *3 mk excursion*



b) *0.5$ excursion*



Reactivity Insertion in a Subcritical Reactor



$$\rho(t) = \begin{cases} \rho_o & t < 0 \\ \rho_o + \Delta\rho & t \geq 0 \end{cases}$$

with:

$$\rho_o = -\frac{s_o}{p_o}$$

- Proceding as previously, we find::

$$\omega_1 = \frac{(a-\lambda)}{2} + \sqrt{\frac{(\lambda-a)^2}{4} + \frac{\lambda\rho}{\Lambda}}$$

$$\omega_2 = \frac{(a-\lambda)}{2} - \sqrt{\frac{(\lambda-a)^2}{4} + \frac{\lambda\rho}{\Lambda}}$$

  - if $\Delta\rho = -\rho_o$, the reactor becomes just critical after the perturbation. In this case, one of the roots is zero:

$$\left.\begin{aligned} \omega_1 &= -\left(\lambda + \frac{\beta}{\Lambda}\right) \\ \omega_2 &= 0 \end{aligned}\right\} \quad \text{if} \quad \rho = 0$$

Subcritical reactor (cont'd)

- if the reactor is <u>not</u> critical after the initial perturbation, we find:

$$p(t) = p_o\left[A_0 + A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t}\right]$$

with

$$A_0 = \frac{\rho_o}{\rho} = \frac{\rho_o}{\rho_o + \Delta\rho}$$

$$A_1 = \frac{(\omega_1 + \lambda)(\omega_1 - \rho_o/\Lambda) + \omega_1 \beta/\Lambda}{\omega_1(\omega_1 - \omega_2)}$$

and

$$A_2 = \frac{(\omega_2 + \lambda)(\omega_2 - \rho_o/\Lambda) + \omega_2 \beta/\Lambda}{\omega_2(\omega_2 - \omega_1)}$$

Approximate expression (since $\lambda\Lambda << \beta$):

$$p(t) \approx p_o\left\{\frac{\rho_o}{\rho} + \left(1 - \frac{\rho_o}{\rho}\right)\left[\left(\frac{\beta}{\beta - \rho}\right) e^{\frac{\lambda\rho}{\beta-\rho}t} - \left(\frac{\rho}{\beta - \rho}\right) e^{\frac{\rho-\beta}{\Lambda}t}\right]\right\}$$

- if the reactor is <u>just</u> critical after the initial perturbation, we find:

$$p(t) \approx p_o\left[1 - \frac{\rho_o}{\beta}(1 + \lambda t - e^{-\frac{\beta}{\Lambda}t})\right]$$

Subcritical Transients

- When a subcritical reactor is perturbed abruptly, there is a prompt jump in neutron power, followed by *a gradual approach to another constant power level.*

- This is a transition between two subcritical states, with the same external source present.

- The new power level is simply:

$$\frac{P_\infty}{P_o} = \frac{\rho_o}{\rho} \quad \Rightarrow \quad \frac{P_2}{P_1} = \frac{\rho_1}{\rho_2}$$

- The measurement of the power change for a know reactivity insertion $\Delta\rho$ allows us to determine the initial reactivity ρ_o.

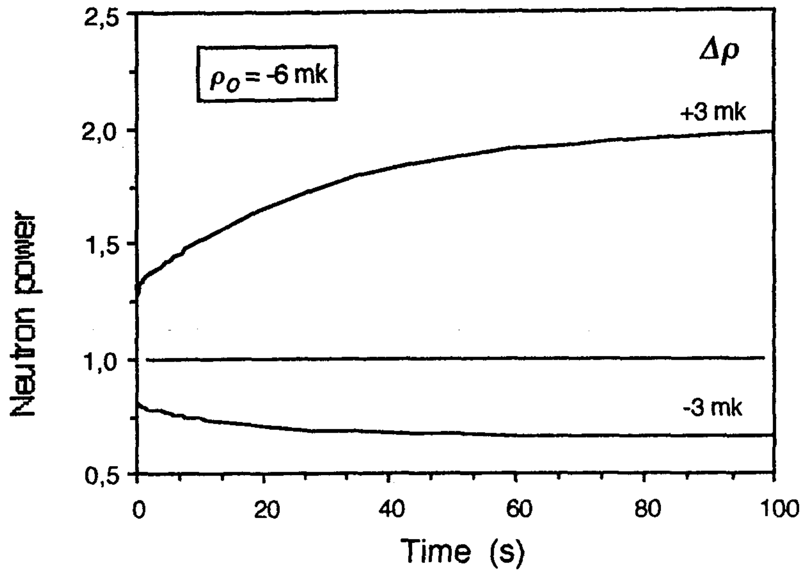
- In particular, when power exactly doubles, the application of an additional $\Delta\rho$ will make the reactor critical (power doubling rule):

  proof:

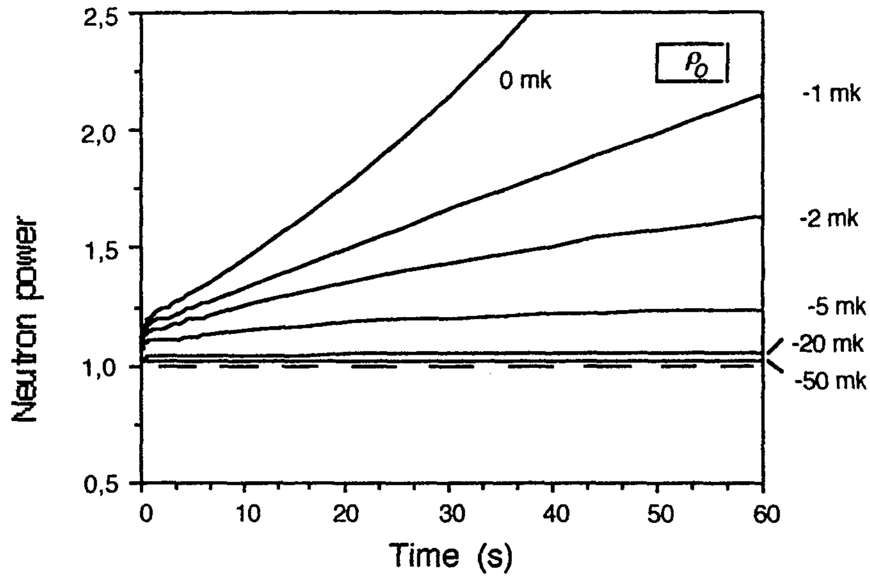
$$\text{if} \quad \frac{P_1}{P_o} = \frac{\rho_o}{\rho} = \frac{\rho_o}{\rho_o + \Delta\rho} = 2$$

$$\text{then} \quad \rho_1 = \rho + \Delta\rho = 0$$

Subcritical Transients in CANDU



1 mk perturbation



Generalization to many delayed neutron groups

With more than one delayed group, we find:

$$\rho(t) = \rho = \begin{cases} \rho_o & t < 0 \\ \rho_o + \Delta\rho & t \geq 0 \end{cases}$$

$$s_d(t) = \sum_{k=1}^{K} \lambda_k \zeta_k$$

$$s(t) = s_o = -\rho_o \, p_o$$

- The solution can be written:

$$\boxed{p(t) = p_o \left[A_0 + \sum_{n=1}^{K+1} A_n \, e^{\omega_n t} \right]} \qquad (\text{if } \rho \neq 0)$$

- The ω_n are obtained from the *Nordheim* equation (also known as the *Inhour* equation):

$$\boxed{\left[\Lambda + \sum_{k=1}^{K} \frac{\beta_k}{\omega + \lambda_k} \right] \omega - \rho = 0}$$

- If the reactor becomes critical after the application of the perturbation,

$$\boxed{p(t) = p_o \left[A_0 + A_1 t + \sum_{n=2}^{K+1} A_n \, e^{\omega_n t} \right]} \qquad (\text{if } \rho = 0)$$

Typical values for CANDU:

a) fresh core:

Groupe k	λ_k (s^{-1})	β_k (%)
1	0,0129	0,0250
2	0,0311	0,1562
3	0,134	0,1505
4	0,331	0,2755
5	1,26	0,0830
6	3,21	0,0317

b) at equilibrium refueling:

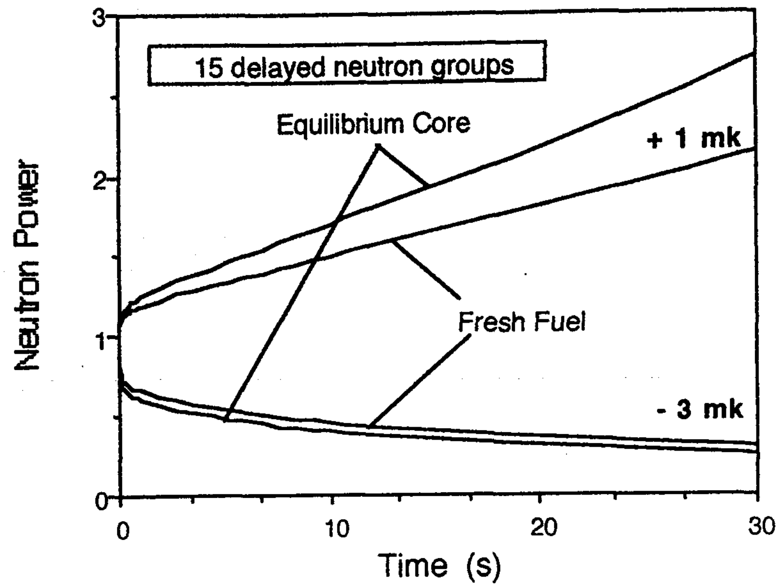
Groupe k	λ_k (s^{-1})	β_k (%)
1	0,0129	0,0189
2	0,0311	0,1245
3	0,134	0,1157
4	0,331	0,2088
5	1,26	0,0658
6	3,21	0,0239

- In addition, 9 groups for photoneutrons are used (see chapter 1)

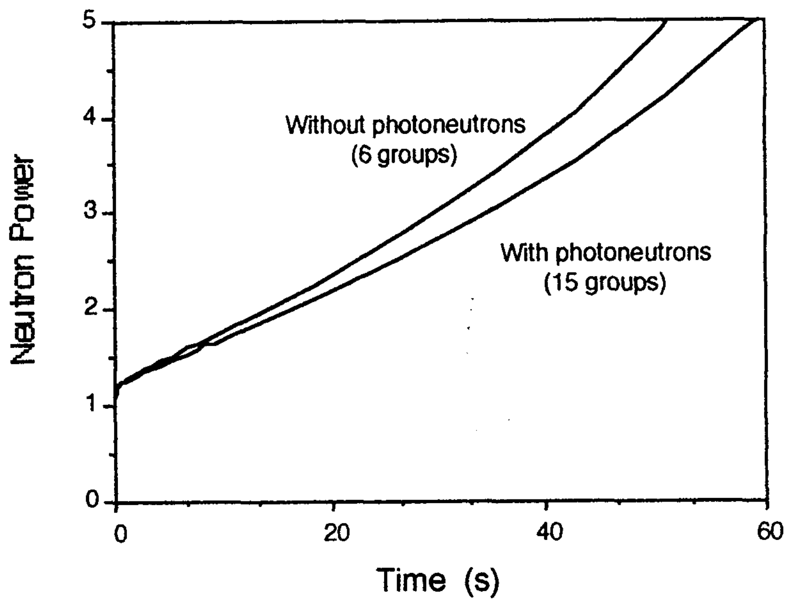
- When photoneutrons are included, the total delayed neutron fraction is:

$$\beta_{fresh} = 0.00755$$
$$\beta_{equil} = 0.00591$$

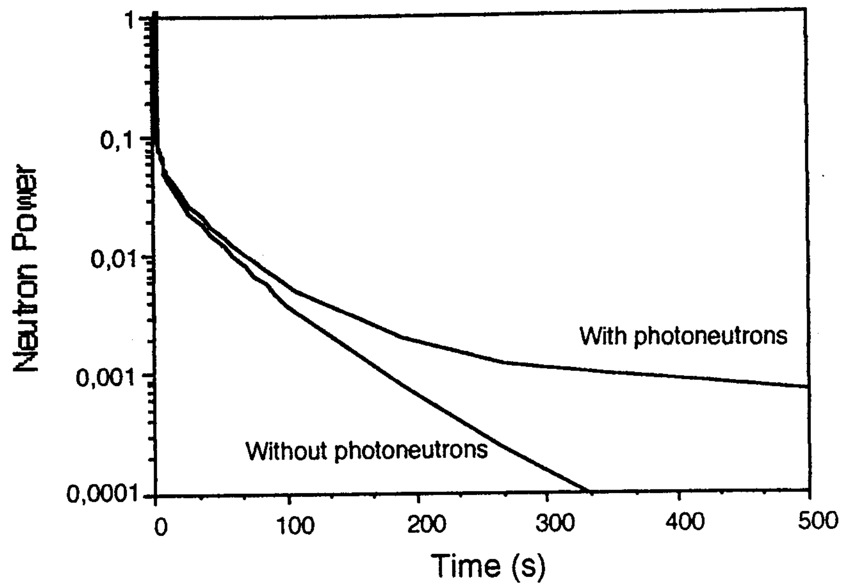
Influence of core composition (CANDU)



Influence of photoneutrons (+1 mk)



Influence of photoneutrons (shutdown, -30 mk)



Comparison with 1 delayed group calculations

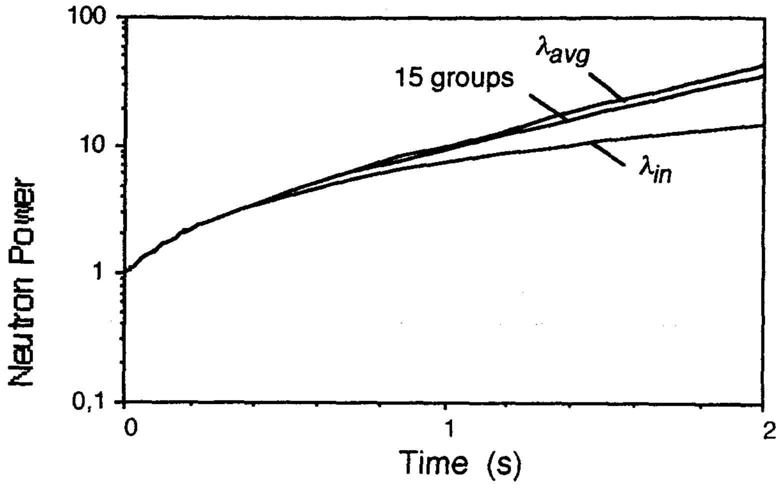
- one-group properties :

$$\beta = \sum_{k=1}^{K} \beta_k$$

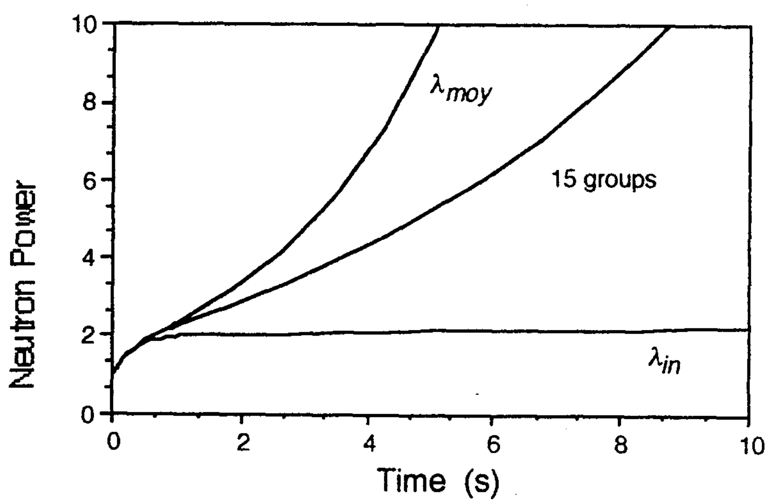
$$\lambda_{avg} = \frac{1}{\beta} \sum_{k=1}^{K} \beta_k \lambda_k$$

$$\lambda_{in} = \frac{\beta}{\sum_{k=1}^{K} \beta_k / \lambda_k}$$

a) +6 mk, equilibrium core



b) +3 mk, equilibrium core



c) -6 mk, equilibrium core