3. The Point Kinetics Equations

- When a perturbation is made to the reactor properties, the steady state no longer holds, and the evolution of the neutron flux must be obtained from the time-dependent diffusion equation.

- The effect of local perturbations on $\phi(\mathbf{r},E,t)$ will quickly propagate. In many cases, there is a slight re-adjustment of the flux shape in a few milliseconds, after which the global flux level will increase or decrease, depending on whether the perturbation has increased or decreased k_{eff}.

- When the flux shape varies little or slowly, one can predict accurately the evolution of the *total reactor power* as a function of the changes brought by the perturbation to the *average* reactor properties, neglecting completely the shape changes. This is *the point-kinetics approximation*.

$$\boxed{\frac{1}{v}\frac{\partial \phi}{\partial t}(\mathbf{r},E,t) = (\mathbf{F}_p - \mathbf{M})\,\phi(\mathbf{r},E,t) + S_d(\mathbf{r},E,t) + S(\mathbf{r},E,t)}$$

with:

$$\mathbf{F}_p\phi = \chi_p(E)\int_0^\infty \nu_p\,\Sigma_f(r,E',t)\,\phi(r,E',t)\,dE'$$

$$\mathbf{M}\phi = -\vec{\nabla}\cdot D\vec{\nabla}\phi + \Sigma_t\phi - \int_0^\infty \Sigma_s(r,E'\rightarrow E,t)\phi(r,E',t)dE'$$

$$S_d = \sum_k \chi_{dk}(E)\;\lambda_k\;C_k(r,t)$$

$$S = \textit{arbitrary independent source}$$

with:

$$\boxed{\frac{\partial C_k}{\partial t} = -\lambda_k C_k(r,t) + \int_0^\infty dE'\;\nu_{dk}\Sigma_f(r,E',t)\phi(r,E',t)}$$

$(k = 1, 2 \ldots K)$

Flux Factorization

- In order to take advantage of the fact that the flux shape can vary more slowly than the amplitude, we introduce the following *factorization*:

$$\phi(r,E,t) = p(t)\ \psi(r,E,t)$$

- Substitution of this factorized form in the diffusion equation leads to a new equation for the shape function Ψ. After dividing by $p(t)$, we find

$$\frac{1}{v}\left(\frac{1}{p(t)}\frac{dp}{dt}\psi(\mathbf{r},E,t) + \frac{\partial\psi}{\partial t}(\mathbf{r},E,t)\right) = \left[\mathbf{F}_p - \mathbf{M}\right]\psi(\mathbf{r},E,t) + \frac{1}{p(t)}S_d(\mathbf{r},E,t)$$

- This equation will be coupled to a second equation for the amplitude $p(t)$.

- This formulation is quite general. The advantage is that the shape Ψ varies much more slowly with t than the amplitude $p(t)$. This permits *an integration step for Ψ much larger than that used to calculate the amplitude p.* The separate solution of the shape and amplitude equations can lead to substantial machine-time savings.

- The equation for the amplitude is obtained by integrating the diffusion equation over the domain $(\mathbf{r},E)$. Since every point in the domain may not have the same importance, we first multiply the diffusion equation by an *arbitrary time-independent weighting function* $w(\mathbf{r},E)$, defined over the same domain as ϕ.

Amplitude Equation

- In analogy with the steady-state production operator, we define the following operator $\boldsymbol{F}$ in the time-dependent case:

$$\boldsymbol{F}\,\phi(\boldsymbol{r},E,t) \;=\; (\boldsymbol{F}_p + \boldsymbol{F}_d)\phi(\boldsymbol{r},E,t)$$

- The operator $\boldsymbol{F}_p$ represents the *instantaneous source of prompt neutrons* (at time t). With the previous factorization, we get

$$\begin{aligned} \boldsymbol{F}_p\,\phi(\boldsymbol{r},E,t) &= p(t)\,\chi_p(E)\int_0^\infty dE'\;\nu_p\,\Sigma_f(\boldsymbol{r},E',t)\;\psi(\boldsymbol{r},E,t) \\ &= p\cdot\boldsymbol{F}_p\,\psi \end{aligned}$$

- The operator $\boldsymbol{F}_d$ *does not* represent the instantaneous source of delayed neutrons. It is the source of delayed neutrons which <u>would</u> be found at equilibrium at $(\boldsymbol{r},E)$ <u>if</u> the reactor were in steady-state with the instantaneous flux $\phi(\boldsymbol{r},E,t)$.

$$\begin{aligned} \mathbf{F}_d\,\phi(\boldsymbol{r},E,t) &= p(t)\sum_k \chi_{dk}(E)\overbrace{\int_0^\infty dE'\;\nu_{dk}\,\Sigma_f(\boldsymbol{r},E',t)\;\psi(\boldsymbol{r},E',t)}^{\text{Spatial distribution of the precursor creation rate}} \\ &= p\cdot\sum_k \mathbf{F}_{dk}\,\psi \\ &= p\cdot\mathbf{F}_d\,\psi \end{aligned}$$

- Substituting in the diffusion equation, we find:

$$\frac{1}{v}\frac{\partial\phi}{\partial t} \;=\; (\boldsymbol{F}-\boldsymbol{M}-\boldsymbol{F}_d)\phi \;+\; S_d \;+\; S$$

- Introducing the factorization $\phi=p\psi$, multiplying by an arbitrary wieght function$w(\boldsymbol{r},E)$ and integrating over the domain, we find the following equation for the amplitude $p(t)$:

$$\left\langle w,\boldsymbol{v}^{-1}\psi\right\rangle\cdot\frac{dp}{dt} + \left\langle w,\boldsymbol{v}^{-1}\frac{\partial\psi}{\partial t}\right\rangle\cdot p \;=\; \left\langle w,(\boldsymbol{F}-\boldsymbol{M}-\boldsymbol{F}_d)\psi\right\rangle\cdot p + \left\langle w,S_d\right\rangle + \left\langle w,S\right\rangle$$

Amplitude Equation (cont'd)

- Delayed-neutron source term contains a weighted sum of precursors:

$$\langle w, S_d \rangle = \sum_{k=1}^{K} \lambda_k \langle w, \chi_{dk} C_k \rangle$$

- pre-multiplying the precursor equation by $\chi_{dk}(E)$, multiplying the result by $w(\boldsymbol{r},E)$, and integrating over the domain, we find:

$$\frac{d\langle w, \chi_{dk} C_k \rangle}{dt} = \langle w, \boldsymbol{F}_{dk}\, \psi \rangle \cdot p(t) - \lambda_k \langle w, \chi_{dk} C_k \rangle$$

- We now define the following scalar quantities to simplify the notation:

$$c_k(t) = \frac{\langle w, \chi_{dk} C_k \rangle}{\langle w, \boldsymbol{v}^{-1} \psi \rangle}$$

$$\xi_\psi(t) = \frac{\left\langle w, \boldsymbol{v}^{-1} \dfrac{\partial \psi}{\partial t} \right\rangle}{\langle w, \boldsymbol{v}^{-1} \psi \rangle}$$

$$s_e(t) = \frac{\langle w, S \rangle}{\langle w, \boldsymbol{v}^{-1} \psi \rangle}$$

- We find:

$$\frac{dp}{dt} = \left(\alpha_\rho(t) - \alpha_\beta(t) - \xi_\psi(t)\right) p(t) + \sum_{k=1}^{K} \lambda_k c_k(t) + s_e(t)$$

and

$$\frac{dc_k}{dt} = \alpha_{\beta_k}(t) \cdot p(t) - \left(\lambda_k + \xi_\psi(t)\right) \cdot c_k(t)$$

$(k = 1, 2 \ldots K)$

Amplitude Equation (cont'd)

- The following *kinetics parameters* have been introduced:

$$\alpha_\rho(t) = \frac{\langle w, (\mathbf{F}-\mathbf{M})\psi \rangle}{\langle w, v^{-1}\psi \rangle}$$

and

$$\alpha_\beta(t) = \sum_{k=1}^{K} \alpha_{\beta_k}(t) = \sum_{k=1}^{K} \frac{\langle w, \mathbf{F}_{dk}\,\psi \rangle}{\langle w, v^{-1}\psi \rangle}$$

- We note that *no approximation has been made to this point.*

- In place of the time-dependent diffusion equation, coupled to the precursor equations, we have after factorization a system of coupled equations consisting of an equation for the shape function and the amplitude equations. <u>These two systems are entirely equivalent</u>.

- In the *quasi-static method*, the shape equation is solved over macro-intervals, within which the amplitude equations are separately integrated, using a known shape. The factorization thus introduces an additional degree of freedom, which can be used to simplify the numerical solution of the time-dependant diffusion equation.

The Normalization Constraint

- We note also that *the relative normalization of p and ψ is arbitrary*, since both these quantities are functions of time. In the absence of a normalization constraint, all solutions of the following form must be allowed:

$$\begin{aligned}\phi(\mathbf{r},E,t) &= p(t)\cdot\psi(\mathbf{r},E,t)\\ &= \left(p(t)\ a(t)\right)\cdot\left(\frac{\psi(\mathbf{r},E,t)}{a(t)}\right)\\ &= \tilde{p}(t)\cdot\tilde{\psi}(\mathbf{r},E,t)\end{aligned}$$

- In order to make the relative normalization unique and to transfer as much as possible of the time variation from ϕ to the amplitude, we will impose:

$$\left\langle w,\mathbf{v}^{-1}\psi\right\rangle = \left\langle\frac{w(\mathbf{r},E)\,\psi(\mathbf{r},E,t)}{v(E)}\right\rangle = K_o$$

where K_0 is an arbitrary constant. As a consequence, we note:

$$\xi_\psi(t) = 0$$

- The amplitude equations then become:

$$\boxed{\frac{dp}{dt} = \left[\alpha_\rho(t) - \alpha_\beta(t)\right]p(t) + \sum_{k=1}^{K}\lambda_k c_k(t) + s_e(t)}$$

and

$$\boxed{\frac{dc_k}{dt} = \alpha_{\beta_k}(t)\cdot p(t) - \lambda_k c_k(t)}$$

(k = 1, 2... K)

The Point Kinetics Approximation

- The normalization constraint does not constitute an approximation:

  The simultaneous solution of amplitude equations together with the above normalization constraint on ψ, affects the value of p but not of the product $\phi = p\psi$, which is conserved.

- The solution of ϕ is independent of the choice for the weighting function w:

  The use of an arbitrary weight function gives an additionnal degree of freedom. We may then select a particular function W to best serve our purpose.

OBSERVATIONS:

- The amplitude equation for $p(t)$ is a set of ordinary differential equations, much easier to solve that the shape equation (for $\psi(t)$) or the original diffusion equation (for $\phi(t)$), which are second order partial differential equations.

- The amplitude equation yielding $p(t)$, a quantity proportional to the total neutron power, is controlled by only two kinetics parameters, namely $\alpha_\rho(t)$ and $\alpha_\beta(t)$, which depend on the *nuclear properties* of materials within the domain.

OBJECTIVE:

Our objective is to decouple the amplitude equation from the shape equation, so that we can estimate simply the *reactor response to localized perturbations.*

The Point Kinetics Approximation (cont'd)

APPROXIMATION:

The central approximation in point kinetics is to assume that the shape function ψ used to calculate the kinetics parameters is constant. It is usually equal the initial steady-state flux distribution.

- The initial steady-state flux distribution, $\phi_o(r,E)$, is normally obtained by solving (once) *the static diffusion equation* with a detailled model of the reactor.

- If, in reality, the localized perturbation does not affect significantly the flux shape, we can the expect that the solution to the point kinetics approximation, $p(t)$, will provide a reasonably accurate prediction of the total reactor power transient.

- With the introduction of this approximation, we now allow a difference between the approximate flux shape, ϕ_o, used to calculate the amplitude, and the real time dependant flux shape, ψ. It will thus be advantageous to choose a weighting function which minimizes the error in $p(t)$ due to the uncertainty in the shape, $\delta\psi = \phi_o - \psi$.

Conventional Form of the Point-Kinetics Equations

(Initially Critical Reactor)

- At initial steady state, we have:

$$\sum_{k=1}^{K} \lambda_k c_k(0) = \frac{\beta}{\Lambda}(0) \cdot p(0)$$

$$\mathbf{M}_0 \phi_0 = \lambda_0 \mathbf{F}_0 \phi_0$$

where, in principle, $\lambda_0 = 1$ since the reactor is assumed critical.

- We shall choose the initial adjoint flux, ϕ_0^*, for the weighting function:

$$w(\mathbf{r},E) = \phi_0^*(\mathbf{r},E)$$

Reason: the error committed *as a result of using the initial shape* to calculate the kinetics parameter is of second order in the shape error, as long as the initial adjoint flux is used as weighting function.

- While the amplitude equation contains in reality only <u>two</u> kinetics parameters, the conventional form of the point-kinetics equations shows <u>three</u> parameters, by explicitly including the *dynamic reactivity* as one parameter in the equation.

- <u>Dynamic reactivity</u> is defined, by analogy to the static reactivity introduced earlier:

$$\rho(t) = \frac{\left\langle \phi_0^*, (\mathbf{F}-\mathbf{M})\phi_0 \right\rangle}{\left\langle \phi_0^*, \mathbf{F}\phi_0 \right\rangle}$$

The Point Kinetics Parameters

- Multiplying and dividing the first kinetics parameter yields:

$$\alpha_\rho(t) = \underbrace{\frac{\left\langle \phi_o^*, (\mathbf{F}-\mathbf{M})\phi_o \right\rangle}{\left\langle \phi_o^*, \mathbf{F}\,\phi_o \right\rangle}}_{\rho(t)} \cdot \underbrace{\frac{\left\langle \phi_o^*, \mathbf{F}\,\phi_o \right\rangle}{\left\langle \phi_o^*, \mathbf{v}^{-1}\phi_o \right\rangle}}_{1/\Lambda(t)}$$

- We have in this way separated the parameter into two components, the dynamic reactivity $\rho(t)$ and a new parameter, $\Lambda(t)$, which we shall call the mean prompt-neutron lifetime:

$$\boxed{\Lambda(t) = \frac{\left\langle \phi_o^*, \mathbf{v}^{-1}\phi \right\rangle}{\left\langle \phi_o^*, \mathbf{F}\,\phi \right\rangle}}$$

- Treating second the parameter in a similar fashion, we find

$$\alpha_{\beta_k}(t) = \underbrace{\frac{\left\langle \phi_o^*, \mathbf{F}_{dk}\,\phi_o \right\rangle}{\left\langle \phi_o^*, \mathbf{F}\,\phi_o \right\rangle}}_{\beta_k(t)} \cdot \underbrace{\frac{\left\langle \phi_o^*, \mathbf{F}\,\phi_o \right\rangle}{\left\langle \phi_o^*, \mathbf{v}^{-1}\phi_o \right\rangle}}_{1/\Lambda(t)}$$

- We have in this way introduced the new parameter $\beta(t)$, which we shall call the effective delayed-neutron fraction:

$$\boxed{\begin{aligned}\beta(t) &= \sum_{k=1}^{K} \beta_k(t) \\ &= \sum_{k=1}^{K} \frac{\left\langle \phi_o^*, \mathbf{F}_{dk}\,\phi \right\rangle}{\left\langle \phi_o^*, \mathbf{F}\,\phi \right\rangle}\end{aligned}}$$

The Point Kinetics Parameters (cont'd)

- Dynamic reactivity $\rho(t)$ has no units. It is generally a small number, so that it is conventional to use the (non-physical) units *mk*, which are simply fractions of 0.001

- $\rho(t)$ varies rapidly and significantly when the reactor is perturbed (control rod movements, refuellings, temperature and density changes,...)

- On the other hand, over the short time horizon of the transients considered in reactor kinetics, we can often neglect the time variation of $\beta(t)$ and $\Lambda(t)$.

- Λ has units of time (s). It measures the length of time it takes for an average neutron to reproduce itself via the fission chain reaction (i.e. the neutron generation)

- β is the *effective* delayed neutron fraction. It is a weighted average and accounts for the non-uniform distribution of fissile material in the reactor and the different delayed neutron emission spectra.

- Typical values:

	CANDU *(heavy water)*	*LWR* *(light water)*	*MAPLE* *(research)*
Λ	Å10^{-3} s	Å10^{-4} s	Å0.7×10^{-4} s
β	Å0.0059	Å0.0075	Å0.0082

The Point Kinetics Equations

- In the absence of an external source, the conventional form of the kinetics equations is therefore the following:

$$\frac{dp}{dt}(t) = \left[\frac{\rho(t)-\beta}{\Lambda}\right] p(t) + \sum_{k=1}^{K} \lambda_k c_k(t)$$

$$\frac{dc_k}{dt}(t) = -\lambda_k c_k(t) + \frac{\beta_k}{\Lambda}\, p(t) \qquad (k = 1, 2 \ldots K)$$

- An alternate form of the P.K. equations can be written, if we choose the following variable for the delayed precursor concentration, instead of $c_k(t)$

$$\zeta_k(t) = \frac{\left\langle \phi_o^*, \chi_{dk} C_k \right\rangle}{\left\langle \phi_o^*, \lambda_o \mathbf{F}_o \psi_o \right\rangle}$$

Then we have:

$$\Lambda \frac{dp}{dt}(t) = [\rho(t)-\beta] p(t) + \sum_{k=1}^{K} \lambda_k \zeta_k(t)$$

$$\frac{d\zeta_k}{dt}(t) = -\lambda_k \zeta_k(t) + \beta_k\, p(t) \qquad (k = 1, 2 \ldots K)$$

Point Kinetics (conclusions)

- The Point Kinetics equations are simple;

- They can be solved analytically in the case of constant reactivity;

- The PK parameters correctly characterize the reactor and can be obtained in a straight-forward manner:

  - rough estimate (survey calculations),

  - detailled estimate starting from a complete description of a reactor and a steady-state solution of the diffusion equation (engineering calculations);

- If reactivity varies (as it generally does during a transient), the PK equations are *easily solved numerically* (on personnal computers);

- Point Kinetics calculation are very useful in **control and safety studies**, because *feedback effects can easily be modelled*;

- Because the *approach is approximate* (assumption of constant flux shape), caution must be used. Validation with detailed space-time kinetics methods (time-dependent diffusion equation in 3D).