Reference: http://epsc.wustl.edu/classwork/454/syllabus.html

1. FUNDAMENTAL CONCEPTS AND MATH REVIEW

1.1. Introduction

Here we provide for your reading pleasure a review of some of the math concepts used in part of this course. Most of this falls under the heading of "**vector calculus**" If you have taken Earth Forces, you have already seen this stuff. You will not be required to manipulate equations in vector calculus, but you will need to understand what the vector calculus operators mean. Think of this as a foreign language.

1.2. Vectors

Scalars (#; magnitude): indicates scale (e.g., mass, temperature, size, density ...)

Vectors (have both direction and magnitude; e.g., velocity, force...):

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \tag{1.1}$$

Here $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are mutually perpendicular unit vectors. In general, a unit vector is given by

$$\mathbf{u}_{\mathrm{v}} = \mathbf{v}/|\mathbf{v}| \tag{1.2}$$

where

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2} \tag{1.3}$$

There are two types of vector products: dot (scalar) product:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{b}|\cos g \tag{1.4}$$

and cross (vector) product:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = |\mathbf{a}||\mathbf{b}|\sin g \tag{1.5}$$

where g is the angle between the two vectors. We shall soon see that there are equivalent operations in calculus, for which we first need to remind ourselves about partial derivatives.

1.3. Partial Derivatives, Gradient Operator, and Taylor Series

1.3.1. Partial Derivatives

There are many instances in science and engineering where a quantity is a function of more than one parameter. Consider a scalar function

$$w = f\left(x, y\right) \tag{1.6}$$

(e.g., let w be the topography, defined at every point on a reference plane; the reference plane is normally sea level). We can make a contour map of this function in the $x - y$ plane. Also, we can take the derivative of the function in any desired direction with vector calculus (= directional derivative). Most important are the partial derivatives, which tell how the function varies with respect to changes in only one of its controlling variables: In the x direction, define:

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f\left(x + \Delta x, y\right) - f\left(x, y\right)}{\Delta x} \tag{1.7}$$

Similarly, in the y direction the derivative is:

$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{f\left(x, y + \Delta y\right) - f\left(x, y\right)}{\Delta y} \tag{1.8}$$

1.3.2 The Gradient

We can use partial derivatives to talk about the change of a function with direction in terms of a vector. For some arbitrary scalar $j\left(x, y, z\right)$ we can define a vector, $\mathbf{F}$:

$$\mathbf{F} = \frac{\partial j}{\partial x}\mathbf{i} + \frac{\partial j}{\partial y}\mathbf{j} + \frac{\partial j}{\partial z}\mathbf{k} \tag{1.9}$$

made up of the components of change of j in the $x, y,$ and z, directions; i.e., the directional derivatives in the $x, y,$ and z directions. In fact, $\mathbf{F}$ gives the magnitude and direction of the maximum change of j The operation on j to obtain $\mathbf{F}$ has a special name: $\mathbf{F}$ is the gradient of j, denoted by

$$\mathbf{F} = \nabla j \tag{1.10}$$

where ∇ is known as the "del" operator (or "that funny upside down triangle"). As a "hungry" operator, we write

$$\nabla = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \tag{1.11}$$

The gradient is an *extremely important* concept in geophysics and hydrology; e.g.,water flow is driven by pressure gradients, the electric field is the gradient of an electrical potential (voltage), gravity is the gradient of a gravitational potential, etc. In general, the potential is the amount of work done to move a test particle (mass, charge, etc.) in a force field.

Finally, consider the surface

$$j\left(x, y, z\right) = c \tag{1.12}$$

It can be shown that ∇j is perpendicular to surfaces of constant j. For example, DC electric fields are perpendicular to surfaces of constant electric potential (voltage).

1.3.1 Taylor Series

One of the handiest topics in applied math is that of a *Taylor Series*. Consider a one-dimensional function $f\left(x\right)$ that is continuous and has all derivatives. Say we know the value of a function at some point, a, but wish to evaluate it at some arbitrary point x. If we know all the derivatives of this function at a:

$$f'(a) \equiv \frac{df\left(x\right)}{dx}\Big|_{x=a} \tag{1.13}$$

and higher derivatives

$$f''(a),\ f'''(a),\ f''''(a), \tag{1.14}$$

etc., then it can be shown that

$$\begin{aligned} f(x) = f(a) + f'(a)\frac{(x-a)^1}{1!} + f''(a)\frac{(x-a)^2}{2!} \\ + f'''(a)\frac{(x-a)^3}{3!} + \ldots \end{aligned} \tag{1.15}$$

and the more terms, the more accurate is the representation. But the larger $(x-a)$, the more terms it will take. Often we are interested in a value of x that is very close to a. Then just the first two terms of the expansion

$$f(x) \approx f(a) + f'(a)(x-a) \tag{1.16}$$

are often adequate, and the representation becomes exact as $x \to a$. We have *linearized* f about a. When f is a function of more than one variable, then we can replace ordinary derivatives with partial derivatives in the Taylor Series expansion.

1.4. The Divergence Operator and Conservation of Mass

1.4.1 Divergence of a vector

The *divergence of a vector* is given by

$$\begin{aligned} \nabla \cdot \mathbf{q} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \left(q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k} \right) \\ &= \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \end{aligned} \tag{1.17}$$

Here we will imagine a fluid flowing through a test volume, and $\mathbf{q}$ will be a volume flow rate [m^3 s^{-1} per unit area (m^2)] so that $\mathbf{q}$ has the dimensions of velocity (m s^{-1}). The divergence of a vector field tests us how much of a field is created or destroyed at any given point. So $\nabla \cdot \mathbf{q}$ tells us about how much a flow rate might change at any given point

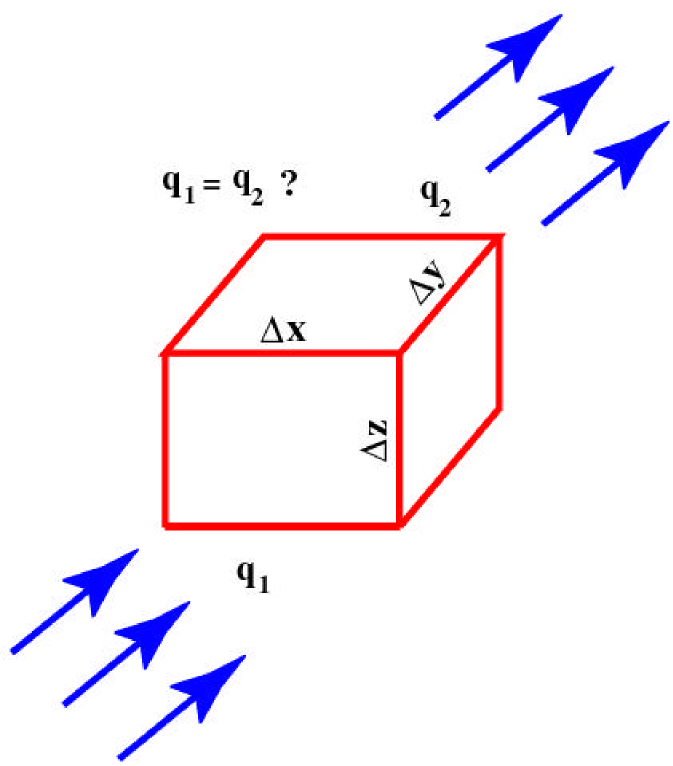

Figure 1.

Consider a fluid flowing through an test cube with dimensions $\Delta x, \Delta y, \Delta z,$ with velocity at the center of the cube given by the components $q_x, q_y,$ and q_z. For the x component of $\mathbf{q}$, at the left wall the mass flow rate is given by

$$\text{left wall:} \quad \left[r_w q_x - \frac{1}{2}\frac{\partial(r_w q_x)}{\partial x}\Delta x \right]\Delta y \Delta z \tag{1.18}$$

(What are the units here?) At the right wall it is

$$\text{right wall:}\quad \left[r_w q_x + \frac{1}{2}\frac{\partial\left(r_w q_x\right)}{\partial x}\Delta x \right]\Delta y \Delta z \tag{1.19}$$

so the gain in mass per unit time in the test box from left to right is

$$\frac{\partial\left(r_w q_x\right)}{\partial x}\Delta x \Delta y \Delta z = \text{L to R gain in mass/s} \tag{1.20}$$

And in the y and z directions it is obviously

$$\begin{gathered} \frac{\partial\left(r_w q_y\right)}{\partial y}\Delta x \Delta y \Delta z \\ \frac{\partial\left(r_w q_z\right)}{\partial z}\Delta x \Delta y \Delta z \end{gathered} \tag{1.21}$$

Thus the total gain in mass <u>per unit volume per unit time</u> is

$$\begin{aligned} &\frac{\left(\frac{\partial\left(r_w q_x\right)}{\partial x}+\frac{\partial\left(r_w q_y\right)}{\partial y}+\frac{\partial\left(r_w q_z\right)}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} \\ &= \frac{\partial\left(r_w q_x\right)}{\partial x}+\frac{\partial\left(r_w q_y\right)}{\partial y}+\frac{\partial\left(r_w q_z\right)}{\partial z} = \nabla\cdot\left(r_w \mathbf{q}\right) \\ &= r_w \nabla\cdot\mathbf{q} \text{ if } r_w \text{ constant} \end{aligned} \tag{1.22}$$

A really handy theorem is the *divergence theorem*

$$\int_{surface} \mathbf{q}\cdot\mathbf{n}\,ds = \int_{volume} \nabla\cdot\mathbf{q}\,dv \tag{1.23}$$

which states that <u>flux</u> of a vector field across a closed surface is equal the divergence of the vector field throughout the enclosed volume.

1.5. Cross Product and Curl

You remember the cross product of two vectors:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = |A||B| \sin\theta \, \mathbf{u_C} \tag{1.24}$$

where $\mathbf{u_C}$ is perpendicular to the plane defined by $\mathbf{A}$ and $\mathbf{B}$, using a right hand system convention (right threaded screw advancing from $\mathbf{A}$ to $\mathbf{B}$ will advance in the direction $\mathbf{C}$). Formally,

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \left(A_y B_z - B_y A_z\right)\mathbf{i} - \left(A_x B_z - B_x A_z\right) + \mathbf{j}\left(A_x B_y - B_x A_y\right)\mathbf{k} \end{aligned} \tag{1.25}$$

An equivalent cross product exists in calculus, called the *curl of a vector*:

$$\begin{aligned} \text{curl } \mathbf{V} \equiv \nabla \times \mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \times \left(u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \right) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ u & v & w \end{vmatrix} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k} \end{aligned} \tag{1.26}$$

A useful theorem is Stokes theorem:

$$\oint_{\Gamma} \mathbf{V} \cdot ds = \int_{A} \left(\nabla \times \mathbf{V} \right) \cdot \mathbf{n} da \tag{1.27}$$

which states that for a curve enclosing an area A, the integral of the projection of a vector onto the curve (“called "the circulation of a vector around the curve”) is equal to the integral of the normal component of the curl of the vector over the area. So the curl tells us something about how a vector field closes or “curls” on itself. Example:

$$\nabla \times H = J \tag{1.28}$$

1.6. Potential fields and the LaPlacian Operator

1.6.1 Potential Fields

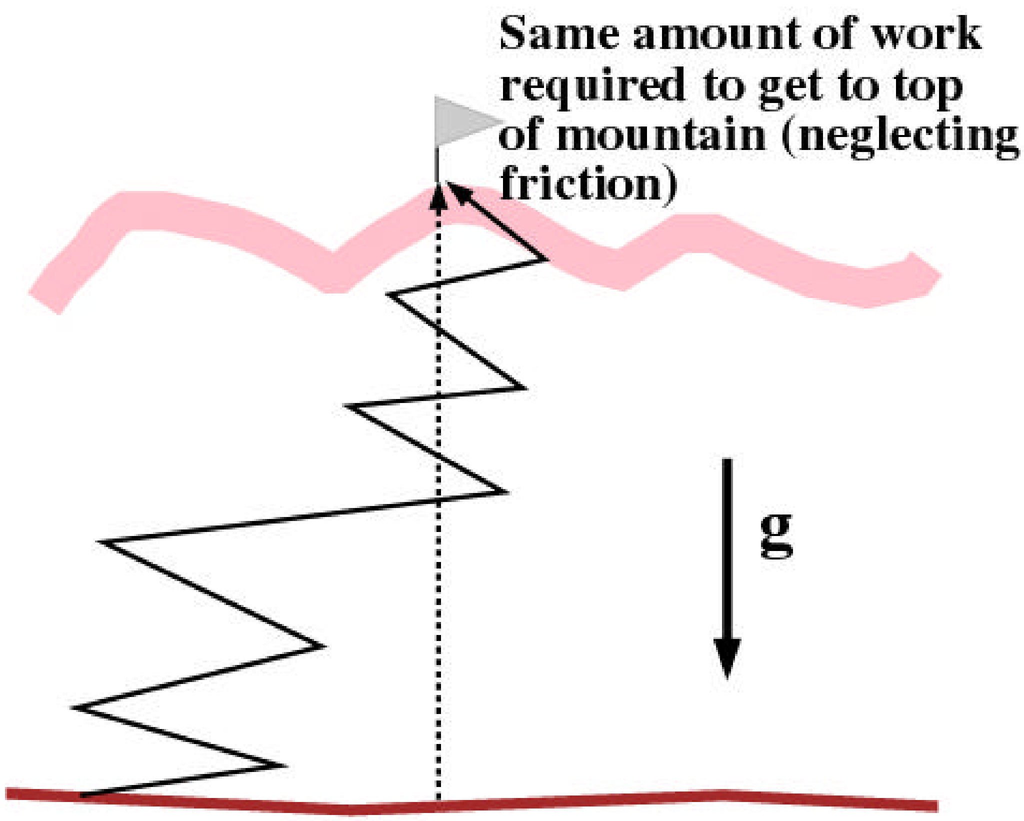

Figure 2.

Many physical fields (<u>what is a field?</u>) in nature have the properties of being potential fields. A potential field is defined as one in which the work (define) required to move around in the presence of the field does not depend on the path we choose to take. The most straightforward example of a potential field is a gravity field.

Other examples of potential fields are DC electric and magnetic fields, the velocity of a fluid when viscosity can be neglected, and hydraulic head.

You recall that work, or energy, equals force times distance:

$$U = \int_1^2 \mathbf{F} \cdot \mathbf{dl} \tag{1.29}$$

A potential field is a force field in which the properties of the field depend only on position -- it does not matter which way the path is taken from “1” to “2” -- the amount of energy required (or given up) is always the same. <u>A fundamental property of a potential field is that</u>

$$\mathbf{F} = -\nabla U \tag{1.30}$$

and this can be shown to result naturally from the independence of path argument. So even though **F** is a vector field, all of its properties are derivable at a point if you use the potential, which is a scalar function, and equation (1.30). You will soon know and love equations like

$$\mathbf{E} = -\nabla V \tag{1.31}$$

where V is the *electrical potential or voltage.* Thus ∇V is the *potential gradient*; the vector **E** is the electric field. One can obviously integrate the electric field between two points to find out the change in potential, and it does not matter what path you take:

$$V = \int_1^2 \mathbf{E} \cdot d\mathbf{l} \tag{1.32}$$

But since we are free to define any vector as a function of a scalar, what is the big deal about potential fields and

$$\mathbf{F} = -\nabla U \tag{1.33}$$

It is this: the relationship in (1.30) <u>completely</u> specifies the physical vector field. Consider a time varying electromagnetic field. The **E** field must be described in terms of both scalar <u>and</u> vector potentials:

$$\mathbf{E} = -\nabla f - \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{\in} \nabla \times \mathbf{A}^* \tag{1.34}$$

where **A** and $\mathbf{A}^*$ are vector potentials of source fields within and external to the regions of interest, respectively.

1.6.2 LaPlacian Operator

Without getting too ahead of ourselves here, if there are no charges in

a volume, then

$$\nabla \cdot \mathbf{J} = 0 \tag{1.35}$$

where **J** is the current density (current per unit cross-sectional area, amps/m^2, in a direction along the normal to the plane containing the area). Ohm's law ($V = IR$) is expressed in terms of the vector fields as

$$\mathbf{J} = s\ \mathbf{E} \tag{1.36}$$

where s is *electrical conductivity*, the reciprocal of resistivity, r. Substituting in terms of the electrical potential:

$$\nabla \cdot \mathbf{J} = \nabla \cdot s\ \mathbf{E} = \nabla \cdot s\ \nabla V = 0 \tag{1.37}$$

and if s is spatially constant, then

$$\nabla \cdot \left(\nabla V\right) \equiv \nabla^2 V = 0 \tag{1.38}$$

where

$$\nabla \cdot \nabla \equiv \nabla^2 \tag{1.39}$$

is the *LaPlacian operator*, which in Cartesian coordinates is :

$$\nabla^2 \equiv \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \tag{1.47}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

Thus (1.38) can be rewritten as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1.48}$$

This is *Laplace's Equation*. In this case it is for electrical potential, but LaPlace's equation appears in many other places; e.g., gravitational potential and hydraulic head obey LaPlace's equation. When there is a time-independent “source term” on the right hand side (i.e., not 0, in the electric field a source of charge or current), then the equation is called Poisson's equation. LaPlace's and Poisson's equations always arise from potential fields and appear when the divergence of the corresponding vector field is taken.