EP 3W4 Final Exam — April 18, 2000

Solution

1. (a) Linear System : one in which the response to a sum of inputs = sum of responses to individual inputs, ie

$$L\left[\sum_n \alpha_n x_n(t)\right] = \sum \alpha_n L[x_n(t)]$$

Implies additive : $L[\sum x] = \sum L[x]$

+ homogeneous : $L(\alpha x) = \alpha L[x]$

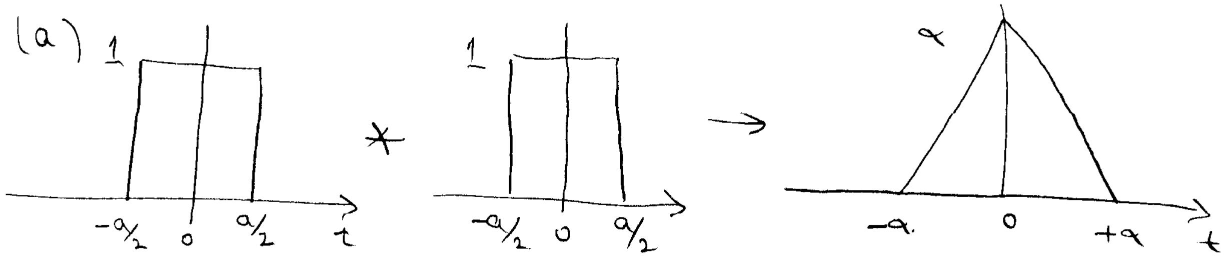
(b) Causal System : response comes after cause.
ie $\int_t^\infty x(\tau)\, h(t-\tau)\, d\tau = 0$

(c) Time Invariant System : response to an input now is the same as the response to a later input.

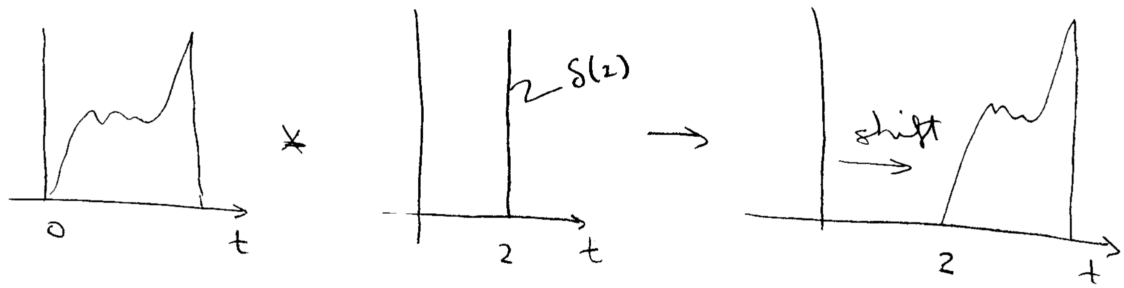
(d) Convolution Integral :
$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$, the response to a input in a linear system is the convolution of the input with the impulse response.

(e) Autocorrelation :
$\int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, d\tau$, a measure of the correlation of one part of a signal to another part of the same signal.

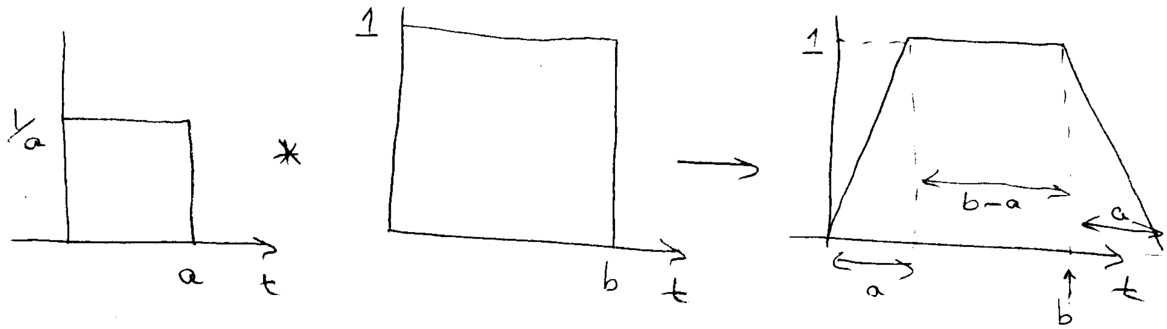
2. (a)


1
-a/2
0
a/2
t
*
1
-a/2
0
a/2
a
-a
0
+a
t

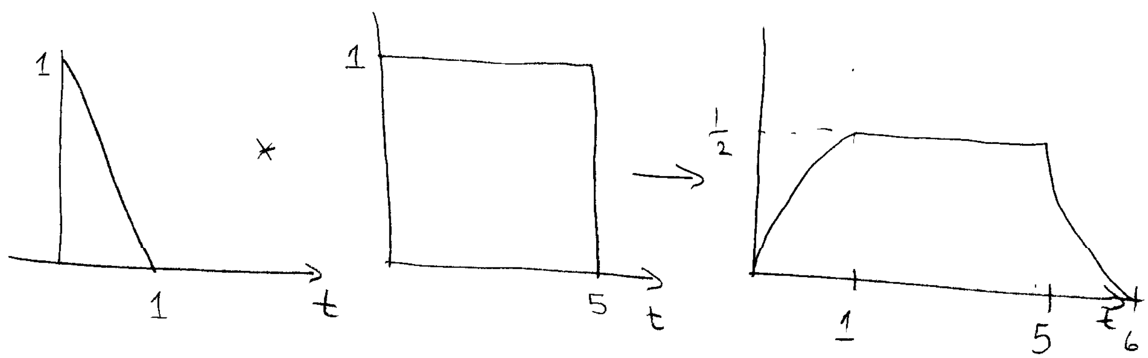
(b)


0
t
*
δ(2)
2
t
shift
2
t

(c)


1/a
a
t
*
1
b
t
1
b-a
a
a
b
t

(d)


1
1
t
*
1
5
t
1/2
1
5
t
6

3. (a) $\displaystyle\int_{-\infty}^{\infty} \delta(-2t-1)\,\mathrm{sinc}(\pi\nu t)\,dt$

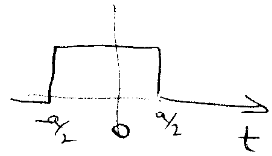
$$= \frac{1}{2}\,\mathrm{sinc}\left[\pi\nu\left(-\frac{1}{2}\right)\right] = \frac{1}{2}\,\mathrm{sinc}\left[\frac{\pi\nu}{2}\right]$$

Since sinc is even

(From $\int f(t)\,\delta(at-b) = \frac{1}{|a|} f(b/a)$

$a = -2$

$b = +1$ $\therefore b/a = \left(-\frac{1}{2}\right)$

(b) $\dfrac{d}{dt}\Pi_a(t) = \dfrac{d}{dt}$  $= \delta(t+a/2) - \delta(t-a/2)$

(c) We know ~~that~~ $\dfrac{df}{dt} \rightleftharpoons 2\pi i\nu F(\nu)$

ie $\dfrac{-i}{2\pi}\dfrac{df}{dt} \rightleftharpoons \nu F(\nu)$

Reverse the role of $\nu + t \Rightarrow \dfrac{-i}{2\pi}\dfrac{dF}{d\nu} \rightleftharpoons t f(t)$

But recall our definition of FT + IFT – they are the same except for $i \to -i$

$\therefore$ when we switch the roles of $t + \nu$ we need to change $i \to -i$.

$\therefore \dfrac{i}{2\pi}\dfrac{dF}{d\nu} \rightleftharpoons t f(t)$ QED

Or We could also substitute $t \to \nu'$, $\nu \to -t'$ as an alternate approach.

Or We could note ~~that~~ if $\dfrac{df}{dt} \rightleftharpoons 2\pi i\nu F(\nu)$, then $\dfrac{dF}{d\nu} \rightleftharpoons -2\pi i t f(t)$ since the only difference in F.T. + I.F.T. is "i" instead of "$-i$". Thus $t f(t) = \frac{1}{2\pi} i \frac{dF}{d\nu}$ directly.

4. (a) When we reflect the time signal about the origin, the FT is even + real. This makes manipulation easier.

(b) We get a convolution integral when we make a measurement in a LTIC system because the response of an instrument (or any LTIC system in general) to an impulse is:

$$y(t) = L[x(t)] = \int_{-\infty}^{\infty} x(\tau)\, L[\delta(t-\tau)]\, d\tau$$

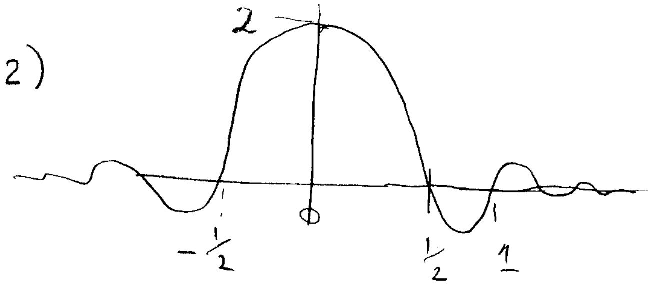
$$\equiv \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

(c) FT's are useful for dealing with convolutions because, in ν space, it is just a multiplication, ie $f * g \rightleftharpoons F \cdot G$

(d) $\mathcal{L}$ works in general due to the e^{-st} factor, ie $\mathcal{L}$ has a real damping component. FT doesnot. So FT is okay for a real signal because it is finite in time + amplitude. We don't always have that in a differential equation so $\mathcal{L}$ is used there. $\mathcal{L}$ can also handle non-zero initial conditions. F.T. cannot.
Also $\mathcal{L}^{-1}$ is easily found by table lookup for known terms in a differential eqn.
There is no known $\mathcal{L}^{-1}$ for a general signal. There is for $\mathcal{F}^{-1}$.

5. (a) $[a * b + cd]^2 \rightleftharpoons [A \cdot B + C * D] * [A \cdot B + C * D]$
where $a \rightleftharpoons A$, etc.

(b) $\Pi_2(t) \rightleftharpoons 2 \operatorname{Sinc}(\pi \nu 2)$



(c) $e^{-\pi t^2} \rightleftharpoons e^{-\pi \nu^2}$

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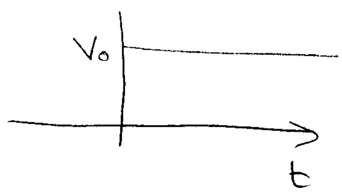
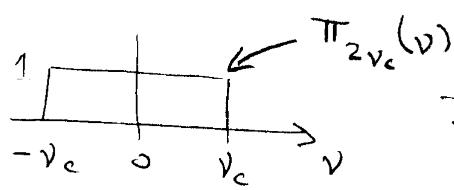
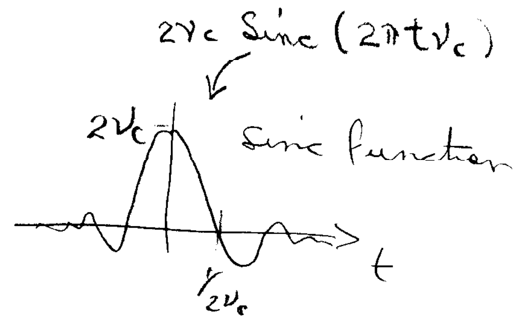
6. (a)



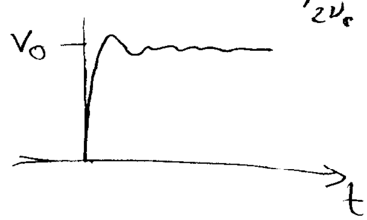
(b)



$\rightleftharpoons$ (c)



(d) $y(t) = V_0\, u(t) * \text{sinc}$



(e) $V_0\, u(t) \rightleftharpoons V_0\, \phi(\nu)$

where $\phi(\nu) = \frac{1}{2\pi i \nu}$ as per the lecture notes.

(f) $Y(\nu) = \underbrace{\frac{V_0}{2\pi i \nu}}_{\text{input}} \underbrace{\Pi_{2\nu_c}(\nu)}_{\text{Filter}}$

g) $$y(t) = \int_{-\nu_c}^{\nu_c} \frac{V_0}{2\pi i \nu} e^{2\pi i \nu t} \, d\nu$$

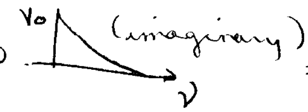
$$= V_0 t \int_{-\nu_c}^{\nu_c} \text{sinc } 2\pi \nu t \, d\nu$$ (see lecture notes page 8-6)

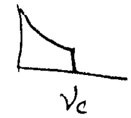
This needs to be numerically integrated + yields the same as (d).

h) Time space:

the input signal is convoluted with the sinc function (which is the ideal filter in t space)

Freq. space:

the high freq. components are filtered out via . Thus 

$\Rightarrow$ 

7. (a) As per lecture notes:

$$\hat{S}\left(\nu+\tfrac{m}{T}\right) = \sum_{n=-\infty}^{\infty} s(nT)\, e^{-2\pi i \nu nT - 2\pi i n m}$$

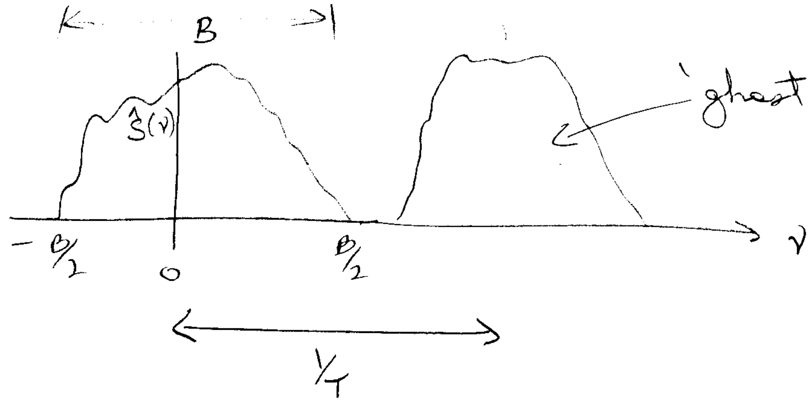
$$= \sum s(nT)\, e^{-2\pi i \nu nT} \cdot \underbrace{e^{-2\pi i n m}}_{=1}$$

since a multiple of 2π

$$= \hat{S}(\nu)$$

$\therefore$ periodic QED

(b)



The sampling rate $1/T$ must be large enough to prevent overlap of the ghost & the signal.

(c) Sampling rate $(1/T)$ must be $\geq 2 \times$ highest frequency $(B/2)$

ie $\frac{1}{T} \geq B$

$\therefore$ critical rate is $\frac{1}{T} = B$ ie, $T = \frac{1}{B}$

— end —