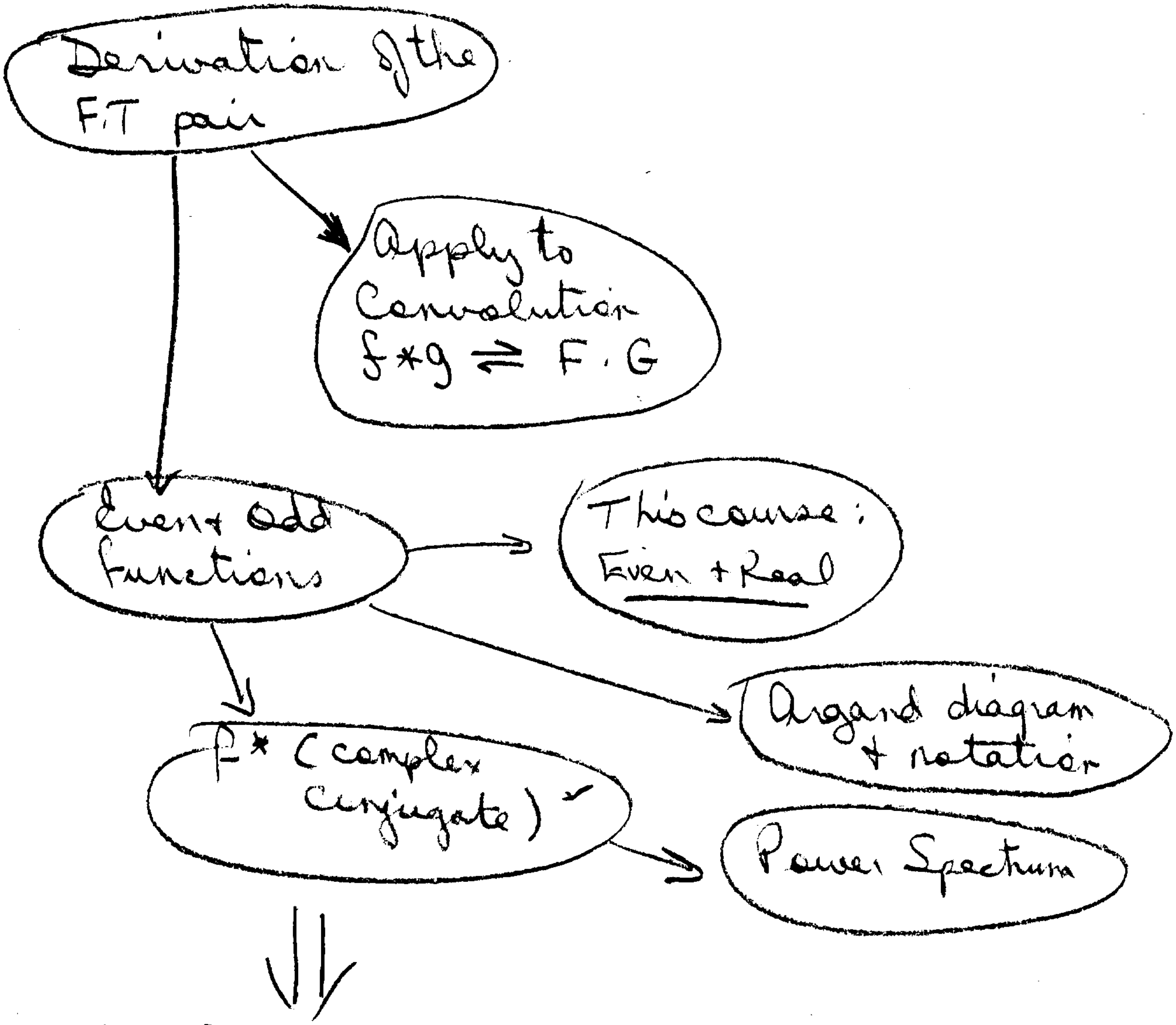
Chapter 5 – Fourier Transforms

5.0 Overview



Useful functions

→ $\Pi_a(t) \rightleftharpoons a \operatorname{sinc}(\pi \nu a)$

→ gaussian $\rightleftharpoons$ gaussian

→ $e^{-t/a} \rightleftharpoons \frac{a}{1+2\pi i \nu a}$

→ $\delta \rightleftharpoons 1$, shifted δ

→ $Ш \rightleftharpoons Ш$

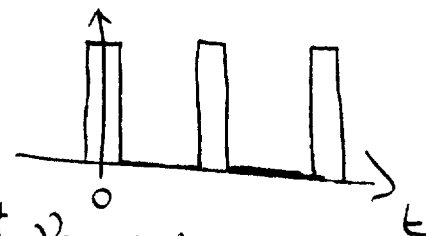
Chapter 5 Fourier Transforms
- Derivation + Useful Functions
(Following James 1.4 → 1.7 except as noted)

5.1 The Fourier Transform (see also Brock 2.4 or Johnston 17.6)

Whether $f(t)$ is periodic or not, we can describe $f(t)$ using sin and cos. We have seen how the Fourier Series can be used for periodic functions. If we let the period go to ∞, we essentially have a non-periodic function.

Consider the square wave where we found

$$f(t) = hb\nu_0 + 2hb\nu_0 \sum_{n=1}^{\infty} \frac{\sin(\pi \nu_0 n b)}{\pi \nu_0 n b} \cos(2\pi n \nu_0 t)$$



We have components (harmonics) at $\nu_0, 2\nu_0, 3\nu_0, \ldots$
If $T (= 1/\nu_0) \to \infty$, then $\nu_0 \to 0$, ie the harmonics get closer and closer together as $T \to \infty$. We have essentially a continuum of frequencies. Thus, we could write that instead of $f(t) = \sum_{n=-\infty}^{\infty} a_n \cos(2\pi n \nu_0 t) + \sum_{n=-\infty}^{\infty} b_n \sin(2\pi n \nu_0 t)$

that we have

$$f(t) = \int_{-\infty}^{\infty} a(\nu) \cos(2\pi\nu t)\, d\nu + \int_{-\infty}^{\infty} b(\nu) \sin(2\pi \nu t)\, d\nu$$

$$\text{or } f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i 2\pi \nu t} d\nu$$

But what is $F(\nu)$?

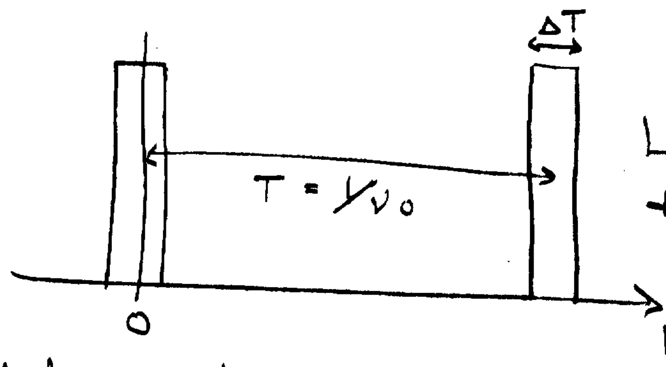
We had $f(t) = \sum_{n=-\infty}^{\infty} D_n e^{2\pi i n \nu_0 t} = \sum_{n=-\infty}^{\infty} D_n e^{2\pi i n t/T}$

where $D_n = \frac{1}{T}\int_T f(t)\, e^{-2\pi i n t/T}\, dt$

← over 1 cycle, ie $\int_{-T/2}^{+T/2}$

Recall: T = period = $1/\nu_0$

We let $T \to \infty$, ie $\frac{1}{T}\,(=\nu_0) \to d\nu$ and $n\,d\nu = \nu$
$= n/T$



Let 'wave' get narrow
+ let period → ∞
∴ wave → impulse

Note: ~~that~~ we straddle the origin. Thus we must reflect $f(t)$ about $t=0$ and make it an even function. This will give negative frequencies, as we shall see.

Thus $f(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-2\pi i n t/T}\, dt \right\} e^{2\pi i n t/T}$

$$\sum_{n=-\infty}^{\infty} \Downarrow \int_{-\infty}^{\infty} \qquad \frac{1}{T} \Downarrow d\nu \qquad \int_{-T/2}^{T/2} \Downarrow \int_{-\infty}^{\infty} \qquad e^{-2\pi i n t/T} \Downarrow e^{-2\pi i \nu t} \qquad e^{2\pi i n t/T} \Downarrow e^{2\pi i \nu t}$$

$$\Rightarrow \int_{-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\, dt \right\}}_{\equiv F(\nu) \,=\, \text{Fourier Transform of } f(t)} e^{2\pi i \nu t}\, d\nu$$

This is the famous Fourier transform pair:

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt \quad \leftarrow \text{note the '$-$' sign.}$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\, e^{i2\pi\nu t}\, d\nu$$

So there you have it.

Now why do we want this? Recall that our original goal was to be able to study system responses, $y(t)$, to inputs, $x(t)$.

We found for linear, time invariant systems,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{(ye ol' Convolution integral)}$$

where h is the system response to an impulse.

We also saw that transforming $y(t)$ via

$$Y(\nu) = \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi\nu t}\, dt \quad \text{gave us}$$

$$Y(\nu) = X(\nu)\, H(\nu)$$

where $X(\nu)$ = Fourier transform of $x(t)$

& $H(\nu)$ = " " " $h(t)$.

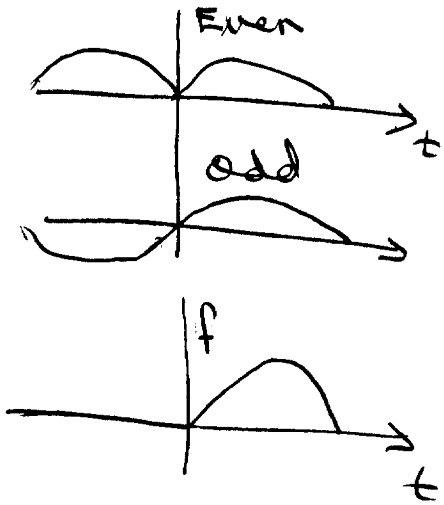
ie, <u>integrals</u> were turned into <u>products</u>

So the general scheme is to formulate our physical system response as an integral in t, transform the parts (x & h) to give X & H, form the product $Y(\nu)$ & transform back to $y(t)$ via

$$y(t) = \int_{-\infty}^{\infty} Y(\nu)\, e^{i2\pi\nu t}\, d\nu.$$

This is a good place to make some statements about even and odd functions and their relationship to real and imaginary numbers in the context of Fourier Transforms.

Any general function, $f(t)$, can be broken up into even, $E(t)$, and Odd, $O(t)$, parts, ie:



$$f(t) = E(t) + O(t)$$

So $$E(t) = \frac{f(t) + f(-t)}{2}$$

+ $$O(t) = \frac{f(t) - f(-t)}{2}$$

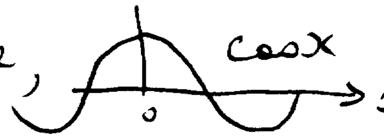
So what do we get when we take the Fourier Transform of $f(t)$?

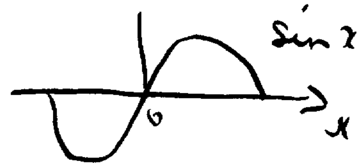
$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \cos(2\pi \nu t)\, dt + i \int_{-\infty}^{\infty} f(t) \sin(2\pi \nu t)\, dt$$

$$= \int_{-\infty}^{\infty} E(t) \cos(2\pi \nu t)\, dt + i \int_{-\infty}^{\infty} E(t) \sin(2\pi \nu t)\, dt$$

$$+ \int_{-\infty}^{\infty} O(t) \cos(2\pi \nu t)\, dt + i \int_{-\infty}^{\infty} O(t) \sin 2\pi \nu t\, dt$$

Now, cos is an even function, ie, 

and sin is an odd function, ie, 

Thus $\int_{-\infty}^{\infty} E(t)\cos(2\pi\nu t)\,dt = 2\int_{0}^{\infty} E(t)\cos(2\pi\nu t)\,dt$.
(ie $\int_{-\infty}^{0} = \int_{0}^{\infty}$)

But $i\int_{-\infty}^{\infty} E(t)\sin(2\pi\nu t)\,dt = 0$ because $\int_{-\infty}^{0} = -\int_{0}^{\infty}$

Likewise $\int_{-\infty}^{\infty} O(t)\cos(2\pi\nu t)\,dt = 0$

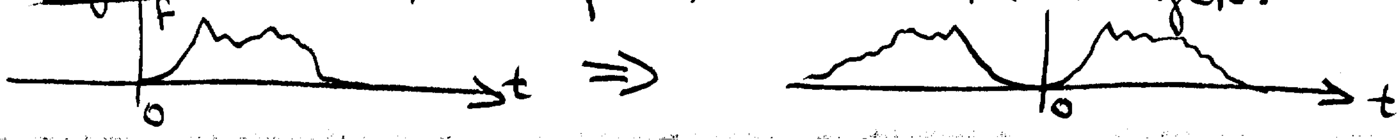
and $i\int_{-\infty}^{\infty} O(t)\sin(2\pi\nu t)\,dt = 2i\int_{0}^{\infty} O(t)\sin(2\pi\nu t)\,dt$

Thus,

$$F(\nu) = 2\int_{0}^{\infty} E(t)\cos(2\pi\nu t)\,dt + 2i\int_{0}^{\infty} O(t)\sin(2\pi\nu t)\,dt.$$

So we will get an <u>imaginary</u> component in the Fourier Transform <u>iff</u> there is an <u>odd</u> component in the input signal, $f(t)$.

Thus, to keep things <u>real</u>, we simply <u>reflect</u> our input signal about the origin.

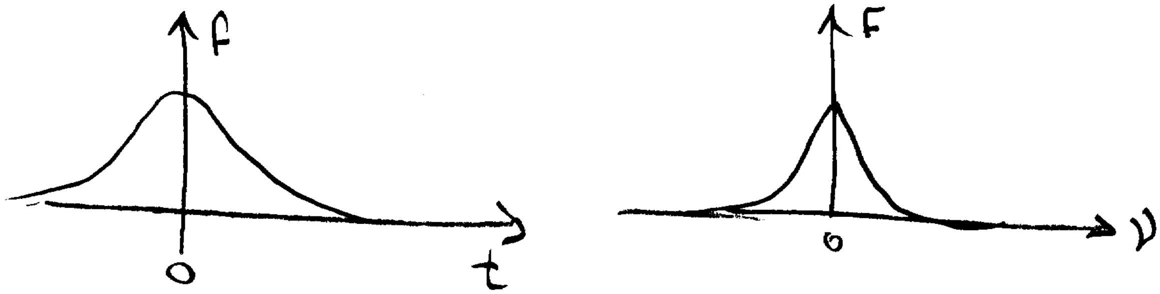


Thus:

$$F(\nu) = 2\int_0^\infty E(t)\cos(2\pi\nu t)\,dt$$
$$= 2\int_0^\infty f(t)\cos(2\pi\nu t)\,dt$$
$$= \text{real}.$$

$F(\nu)$ is also even since cos is even, hence $F(\nu) = F(-\nu)$.

So in this course we will only deal with $f(t)$ + $F(\nu)$ that are real and even.



5.2 Conjugate variables (following James 1.5)

We have been using t and ν as our variables. In general, the literature may use x + p as the transform pairs. Whatever, the units of the products are always dimensionless.

$\left\{\begin{matrix} x + p \\ t + \nu \end{matrix}\right\}$ are called 'conjugate variables'

Power Spectrum

If we had a voltage, $V(t)$, then for a unit resistor ($1\,\Omega$), the power is

$$V^2 R = V^2.$$

In general V may be complex so we write the power density as $V(t)\,V^*(t) = |V(t)|^2$.

In the transformed domain we have

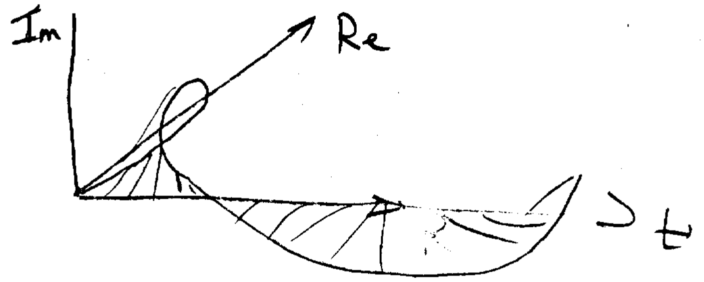
$$S(\nu) \equiv \Phi(\nu)\,\Phi^*(\nu) = |\Phi(\nu)|^2$$ as the power per unit frequency.

This is often called the Power Spectrum or Spectral Power Density (SPD).

For example, optical spectrometers would measure SPD.

5.3 Graphical representation (following James 1.6)

Generally, when a real $F(x)$ fn is transformed, we get a complex fn. $\Phi(\nu)$. To plot $\Phi(\nu)$ we use an Argand diagram.



Note: at this point James switches to x & p

where $\Phi(p) = \int_{-\infty}^{\infty} F(x) e^{2\pi i px} dx$

$$F(x) = \int_{\infty}^{\infty} \Phi(p) e^{-2\pi i p x} dx$$

Where the + & - signs are switched from the t & ν notation.

I'll stick with t & ν and the form we derived;

$$\Phi(\nu) = \int_{-\infty}^{\infty} F(t) e^{-2\pi i \nu t} dt$$

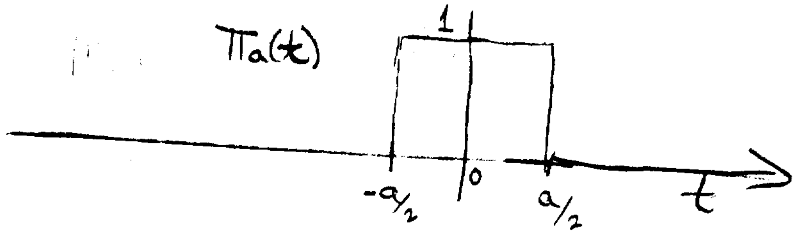
$$F(t) = \int_{-\infty}^{\infty} \Phi(\nu) e^{2\pi i \nu t} d\nu$$

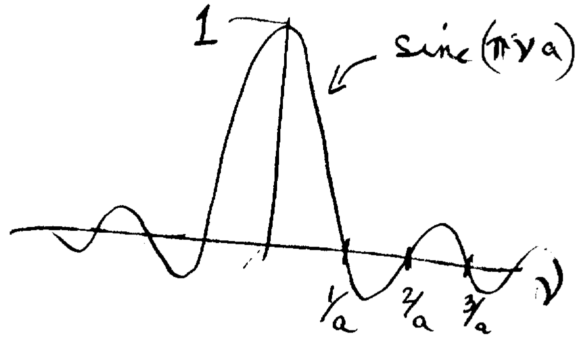
5.4 Useful functions (following James 1.7)

You will see some functions time and time again so, like the multiplication tables, you might as well take note. Don't dispair; with a bit of practise, this will be old hat.

5.4.1 The 'top-hat' function $\Pi_a(t)$

Speaking of hats, lets define the 'top hat' function (a.k.a. a 'box-car' but more commonly as the 'rect' (for rectangular) function.





Thus,

$$\Phi(\nu) = \int_{-\infty}^{\infty} \Pi_a(t)\, e^{-2\pi i \nu t}\, dt$$

$$= \int_{-a/2}^{a/2} e^{-2\pi i \nu t}\, dt$$

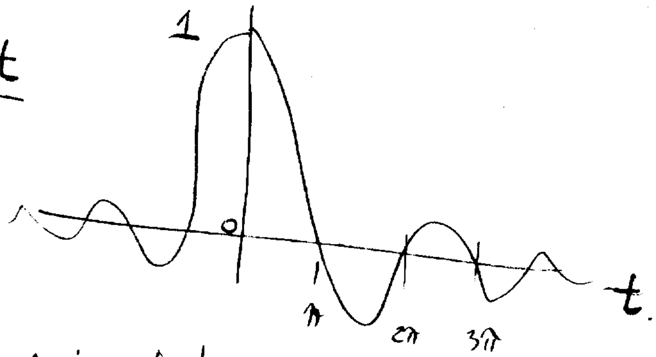
$$= \left. \frac{e^{-2\pi i \nu t}}{-2\pi i \nu} \right|_{-a/2}^{a/2} = \frac{e^{-\pi i \nu a} - e^{\pi i \nu a}}{-2\pi i \nu}$$

$$= a\,\frac{\sin(\pi \nu a)}{(\pi \nu a)} = a \operatorname{sinc}(\pi \nu a)$$

Thus $\quad \Pi_a(t) \rightleftharpoons a \operatorname{sinc}(\pi \nu a)$

5.4.2. The sinc-function

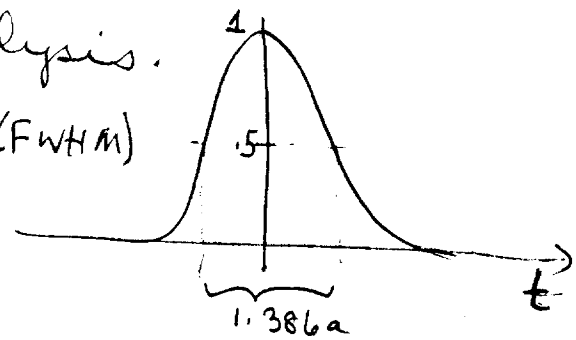
$$\mathrm{sinc}(t) = \frac{\sin t}{t}$$



It has zeros where $\sin t$ has zeros – ie at $n\pi$.
De l'Hôpital's rule gives $\frac{\sin 0}{0} = 1$.

5.4.3 The Gaussian function

The Gaussian is defined as $g(t) = e^{-t^2/a^2}$.
You've seen it in statistical analysis.
It has a Full Width at Half Maximum (FWHM) of $1.386a$.



Furthermore,

$$\int_{-\infty}^{\infty} e^{-t^2/a^2} dt = a\sqrt{\pi}$$

Since a Gaussian is a common ensemble of input signals, we'll need to know the transform.

$$G(\nu) = \int_{-\infty}^{\infty} e^{-t^2/a^2} e^{-2\pi i \nu t} dt$$

Now,

$$t^2/a^2 + 2\pi i \nu t = (t/a + \pi i \nu a)^2 - \underbrace{(\pi i \nu a)^2}_{= +(\pi \nu a)^2}$$

$$\therefore G(\nu) = \int_{-\infty}^{\infty} e^{-(t^2/a^2 + 2\pi i \nu t)}\, dt = e^{-(\pi \nu a)^2} \int_{-\infty}^{\infty} e^{-(t/a + \pi i \nu a)^2}\, dt$$

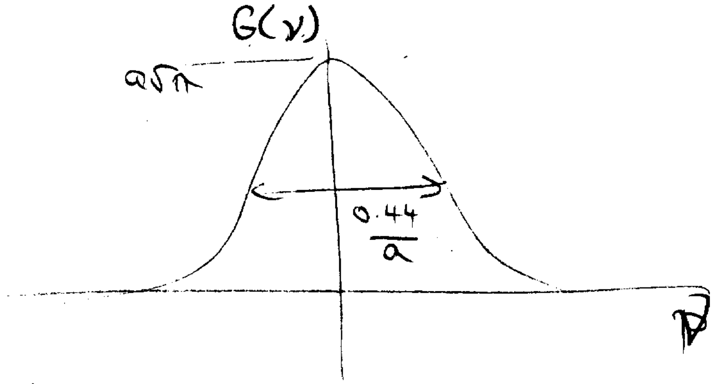
Let $z = t/a + \pi i \nu a \Rightarrow a\,dz = dt$

$$\therefore G(\nu) = a e^{-(\pi \nu a)^2} \underbrace{\int_{-\infty}^{\infty} e^{-z^2}\, dz}_{= \sqrt{\pi}}$$

$$\therefore \boxed{G(\nu) = a\sqrt{\pi}\, e^{-\pi^2 \nu^2 a^2}}$$

This is another Gaussian (of width $\frac{1}{\pi a} = \frac{0.44}{a}$)

Also note at $G(0) = a\sqrt{\pi}$ which is the area under $g(t)$.

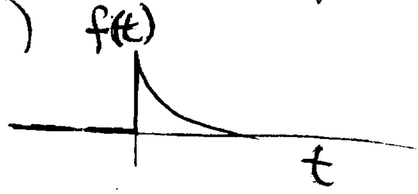


<u>Note</u>: The value of any Fourier Transform evaluated at $\nu = 0$ is the area under the original signal, ie:

$$F(0) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i 0 t}\, dt = \int_{-\infty}^{\infty} f(t)\, dt$$

5.4.4 The exponential decay

Another common signal is a decaying one (electronic, radioactive ...)



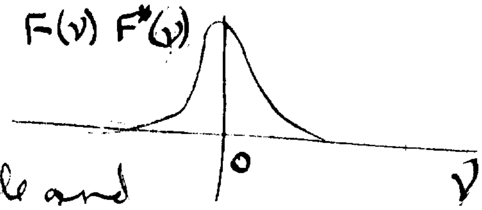
$$f(t) = e^{-t/a},\ t \geq 0$$
$$= 0 \quad t < 0$$

$$\therefore\ F(\nu) = \int_0^\infty e^{-t/a} \cdot e^{-2\pi i \nu t}\, dt$$

$$= \left. \frac{e^{-(t/a + 2\pi i \nu t)}}{-(1/a + 2\pi i \nu)} \right|_0^\infty = \frac{0 - 1}{-(\quad)}$$

$$= \frac{1}{1/a + 2\pi i \nu} = \frac{a}{1 + 2\pi i \nu a}$$

$$\text{\&}\ F(\nu) F^*(\nu) = \frac{a^2}{1 + 4\pi^2 \nu^2 a^2} \rightarrow$$



This is called a Lorentz Profile and is found in spectrum lines at very low pressure. The FWHM, $\Delta\nu$, is $\frac{1}{\pi a}$ and is related to the excited state lifetime. Thus, the Lorentz profile is used to determine atomic transition times.

(Notice ~~that~~ $f(t)$ is not an even function and the subsequent $F(\nu)$ is complex)

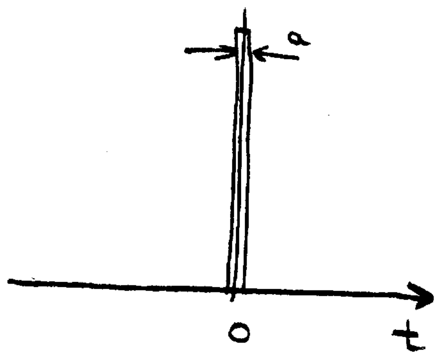
5.4.5 The Dirac 'delta-function'

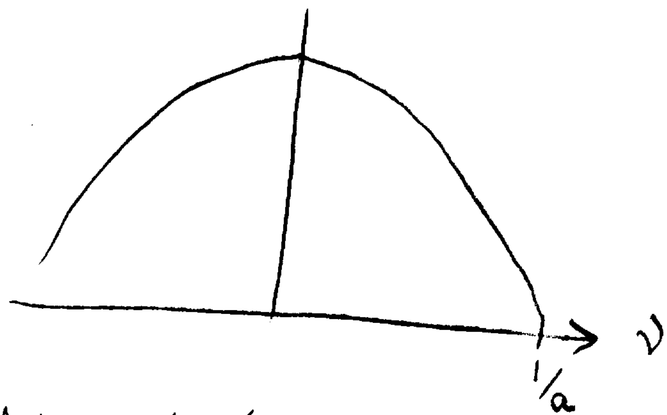
We have seen this before:

$$\delta(t) = 0 \quad \text{for } t \neq 0$$
$$= \infty \quad \text{for } t = 0$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

The δ function is not a well behaved function – it goes to ∞ at $t=0$ but its integral is well behaved.

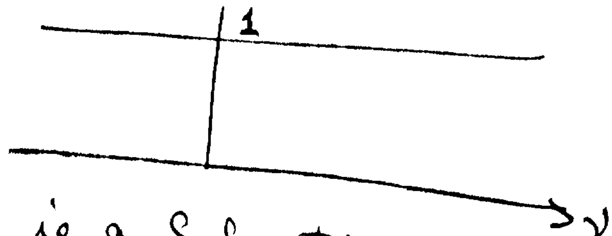
We can determine its transform by taking the limiting case of the Π function:





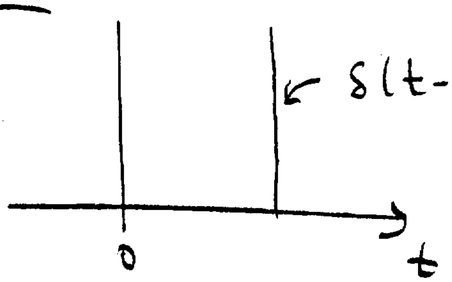
$$\lim_{a\to 0} \frac{1}{a}\Pi_a(t) \rightleftharpoons \lim_{a\to 0} \operatorname{sinc}(\pi\nu a) = \lim_{a\to 0} \frac{\sin(\pi\nu a)}{\pi\nu a}$$

$$\Downarrow$$



Hence $\delta(t) \rightleftharpoons 1$ ie a δ function contains <u>all</u> frequencies

Also



& $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$

Thus:

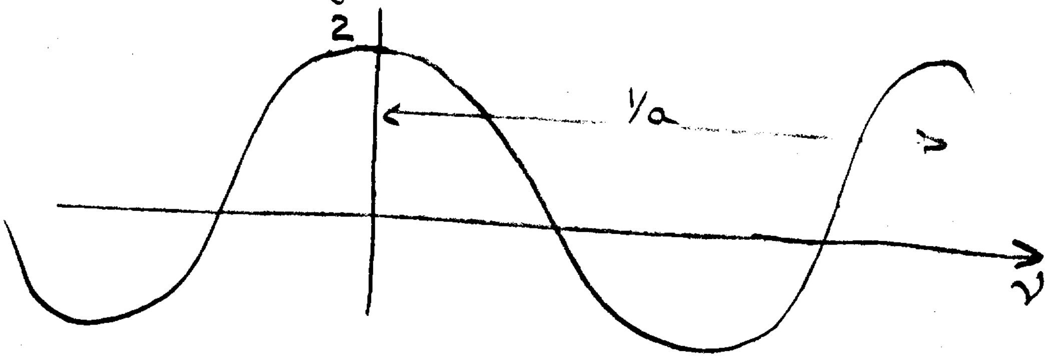
$$\int_{-\infty}^{\infty} e^{-2\pi i \nu t}\,\delta(t-a)\,dt = e^{-2\pi i \nu a}$$

$$\therefore\ \delta(t-a) \rightleftharpoons e^{-2\pi i \nu a}$$

& hence $\delta(t-a) + \delta(t+a) \rightleftharpoons e^{-2\pi i \nu a} + e^{2\pi i \nu a}$

$$= 2\cos(2\pi \nu a)$$

The 2 δ functions combined give constructive and destructive interference.



The Dirac Comb

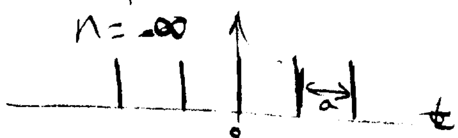
For a large collection of equally spaced δ functions, we define the shah function:

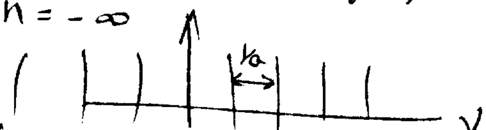
$$Ш_a(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t-na)$$

It turns out that the Fourier Transform of a shah function is another shah function!

$$Ш_a(t) \rightleftharpoons \frac{1}{a}\, Ш_{1/a}(\nu)$$

$$\sum_{n=-\infty}^{\infty} \delta(t-na) \rightleftharpoons \sum_{n=-\infty}^{\infty} \delta\left(\nu-\frac{n}{a}\right)$$





When I find an understandable proof, I'll let you know.

This function will turn out to be useful as a replicating tool.

5.5 Recap

- Fourier Transform Pair:

$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t}\, dt$$

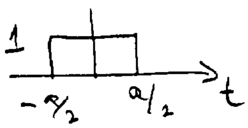
$$f(t) = \int_{-\infty}^{\infty} F(\nu)\, e^{2\pi i \nu t}\, d\nu$$

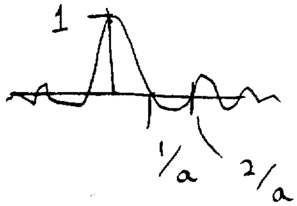
If $f(t)$ is <u>even</u> then $F(\nu)$ is real.

$\Phi(\nu)\, \Phi^*(\nu) = |\Phi(\nu)|^2$ is the power per unit frequency (Power Spectrum or Spectral Power Density (SPD)).

Rect:

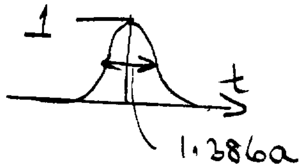
$$\Pi_a(t) \rightleftharpoons a\,\mathrm{sinc}(\pi \nu a)$$

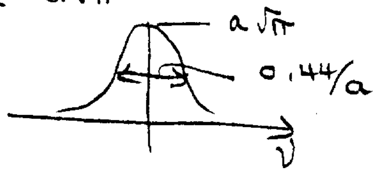




Gaussian: $g(t) = e^{-t^2/a^2}$, $\int_{-\infty}^{\infty} e^{-t^2/a^2} = a\sqrt{\pi}$

$$g(t) \rightleftharpoons a\sqrt{\pi}\; e^{-\pi^2 \nu^2 a^2}$$





Exponential decay

$$e^{-t/a} \rightleftharpoons \frac{a}{1+2\pi i \nu a}$$

Delta:

$$\delta(t) \rightleftharpoons 1 \quad , \quad \delta(t-a) \rightleftharpoons e^{-2\pi i \nu a}$$

$$\delta(t-a) + \delta(t+a) \rightleftharpoons 2\cos(2\pi \nu a)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = f(a)$$

Dirac Comb:

$$Ш_a(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t-na)$$

$$Ш_a(t) \rightleftharpoons \frac{1}{a}\, Ш_{1/a}(\nu)$$