Chapter 4: Complex Numbers (Following Johnson Appendix C + D)

We have introduced complex numbers when we wrote: $e^{i\theta} = \cos\theta + i\sin\theta$.

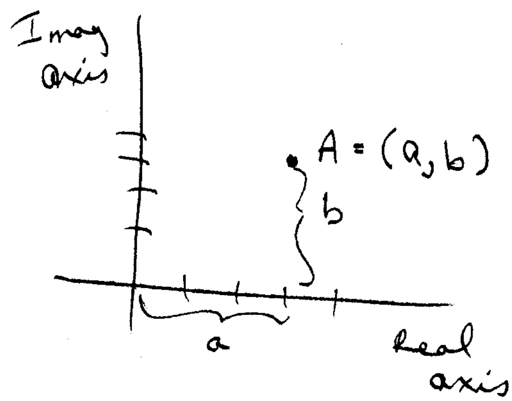
The payoff for doing this is the ease of manipulation of Fourier Transforms. The price we pay is the disconnection from reality; imaginary numbers are, well, unreal! It is, thus, worthwhile reviewing complex numbers so we understand what they mean in a physical sense – we need to keep our link to reality if we are to employ F.T. to real, physical systems.

We define i (or j in electrical engineering) $= \sqrt{-1}$

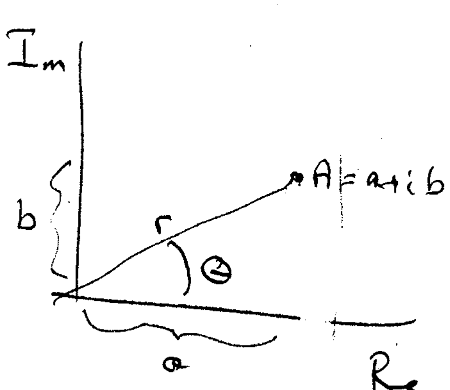
+ say ~~that~~ $A = \underbrace{a}_{\text{real part}} + i\underbrace{b}_{\text{imaginary part}}$ is a complex number.

← this is a poor choice of words.

We represent a complex number using Cartesian coord;



or in terms of r, Θ (polar coordinates)



Thus $A = 4 + i\,3 = 5\angle 36.9°$ ⟵

$r = \sqrt{a^2+b^2} = |A|$

$\Theta = \tan^{-1}\dfrac{b}{a}$

angle argument $\equiv$ ang A = arg A

$a = r\cos\Theta$

$b = r\sin\Theta$

$A = r\angle\Theta$

Complex conjugate: $A^* = a - ib$

$$\therefore |A^*| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2} = |A|$$

$$\arg A^* = \tan^{-1}\left(\frac{-b}{a}\right) = -\tan^{-1}\left(\frac{b}{a}\right) = -\arg A$$

$$\therefore (r\angle\theta)^* = r\angle{-\theta}$$

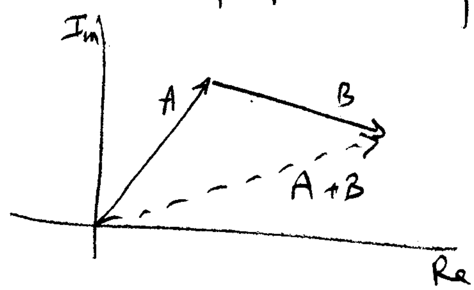
But how do we manipulate complex numbers? Is it valid to use a Cartesian form?

<u>Addition & Substraction</u>:

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

$\therefore$ addition (& substraction) works.

Can do this graphically;



Multiplication

$$A = a + ib = r_1 \cos\theta_1 + i r_1 \sin\theta_1$$

$$B = c + id = r_2 \cos\theta_2 + i r_2 \cos\theta_2$$

$$\therefore AB = (a+ib)(c+id) = ac + iad + ibc + i^2 bd$$

$$= (ac - bd) + i(ad + bc) \qquad \text{Yuck!}$$

Let's try the polar form:

$$AB = (r_1 \cos\theta_1 + i r_1 \sin\theta_1)(r_2 \cos\theta_2 + i r_2 \sin\theta_2)$$

$$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right]$$

$$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

$$\text{ie } AB = (r_1 \angle\theta_1)(r_2 \angle\theta_2) = \underbrace{r_1 r_2}_{\text{multiply the lengths}} \ \& \ \underbrace{\angle \theta_1 + \theta_2}_{\text{add the angles.}} \quad \leftarrow \text{Better!}$$

From this we see ~~that~~

$$AA^* = (r\angle\theta)(r\angle-\theta) = r^2 \angle 0 = |A|^2 \angle 0 = |A|^2$$

a real number.

$$\text{ie} \quad AA^* = |A|^2$$

Division

$$N = \frac{A}{B} = \frac{a+ib}{c+id}$$

We rationalize the denominator via

$$N = \frac{A}{B} = \frac{AB^*}{BB^*} = \frac{(a+ib)\ (c-id)}{(c+id)\ (c-id)}$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$ Yuck!

Better to use the polar form:

$$N = \frac{A}{B} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1}{r_2} \underline{/\theta_1 - \theta_2}$$

So better to add + subtract in rectangular form + multiply + divide in polar form.

Real ≠ Imaginary

Maybe this is stating the obvious but if

$$a + ib = c + id$$

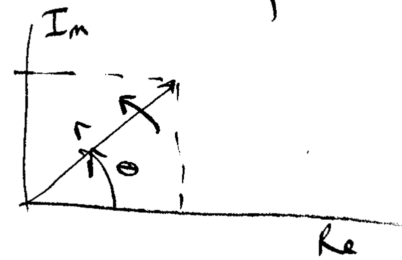
then $a - c = i(d - b)$.

Since a real ≠ an imaginary, we must have

$$a = c \ + \ d = b.$$

Euler's Formula

We said ~~that~~ $e^{i\theta} = \cos\theta + i\sin\theta$. This is Euler's Formula. This is a rather amazing statement when you think about it. Why should an anti-log be related so wonderously to trig functions? And why does it have such a beautiful graphical representation of a spinning vector?



To derive Euler's Formula, let

$$g = \cos\theta + i\sin\theta \qquad (\theta = \text{real})$$

Thus
$$\frac{dg}{d\theta} = -\sin\theta + i\cos\theta$$
$$= i^2\sin\theta + i\cos\theta$$
$$= i(\cos\theta + i\sin\theta) = ig.$$

Thus
$$\frac{dg}{g} = i\,d\theta \Rightarrow \ln g = i\theta + \text{constant}$$

But $g(\theta) = 1$ when $\theta = 0$

$$\therefore \ln g(0) = \ln 1 = 0 = 0 + \text{constant}$$

$$\therefore \text{constant} = 0.$$

$$\therefore g = e^{i\theta}$$
$$= \cos\theta + i\sin\theta \qquad \text{Q.E.D.}$$

Now $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$$= \cos\theta - i\sin\theta$$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\& \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This is all very useful.

Now let's consider the polar form of $A\ (= a+ib)$

$$a = r\cos\theta \qquad \& \ A = r\angle\theta$$
$$b = r\sin\theta$$

$$\therefore re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$= a + ib = r\angle\theta$$

Wonderful! The polar form of A is just $re^{i\theta}$.
So to manipulate A, we just manipulate $re^{i\theta}$.
Thus multiplication becomes:

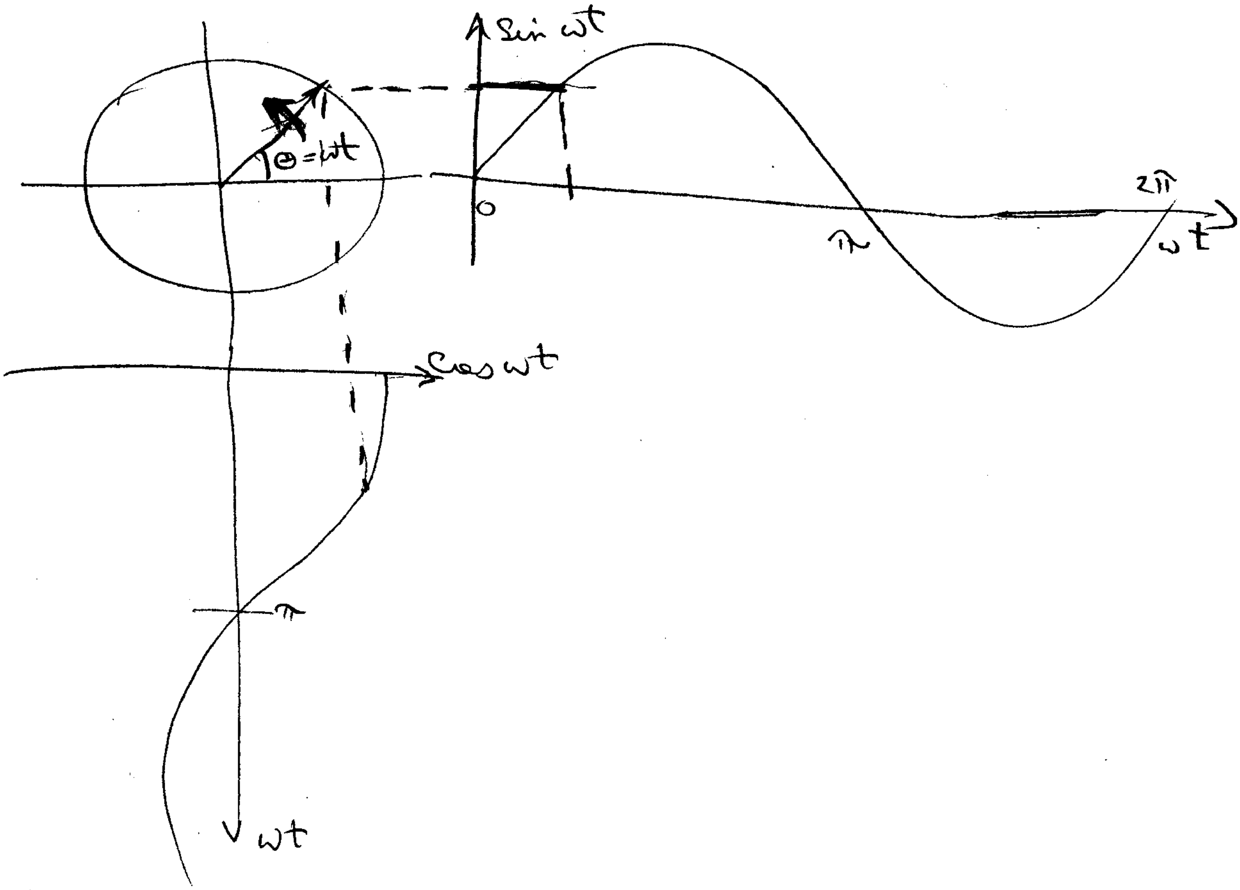
$$AB = (r_1 e^{i\theta_1})\, r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}$$

& Division becomes

$$A/B = r_1/r_2 \; e^{i(\theta_1-\theta_2)}$$

Don't you love it when a plan comes together!

If we consider the angle Θ as denoting a rotating vector of angular freq. ω $(= 2\pi\nu)$, then $\Theta = \omega t$
& $e^{i\Theta} = e^{i\omega t} = e^{i 2\pi\nu t} = \cos \omega t + i \sin \omega t$



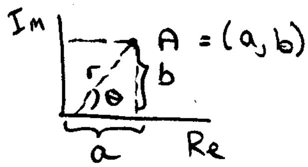
This is the Phasor notation, which derives its name from the fact that we are just talking about the relative angles or phases between the spectral components. The sin wave is no more unreal than the cosine; complex numbers and the "imaginary" notation are just a convenient representation of very real phenomena.

Recap

$i \equiv \sqrt{-1}$

$A = a + ib$

$\quad = r \angle \theta$

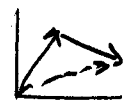


$A^* = a - ib$

$\quad =$ complex conjugate

$|A^*| = |A| \;, \quad AA^* = |A|^2$

Can vector add and subtract

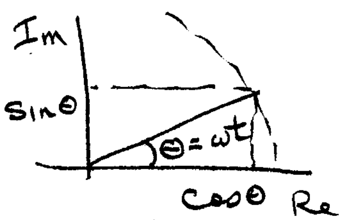


$AB = r_1 r_2 \; \angle(\theta_1 + \theta_2)$

$A/B = r_1/r_2 \; \angle(\theta_1 - \theta_2)$

$A \pm B = (a \pm c) + i(b \pm d)$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)



← Phasor notation

$\therefore \quad A = a + ib = re^{i\theta} = r\angle\theta$

Useful for manipulating complex numbers