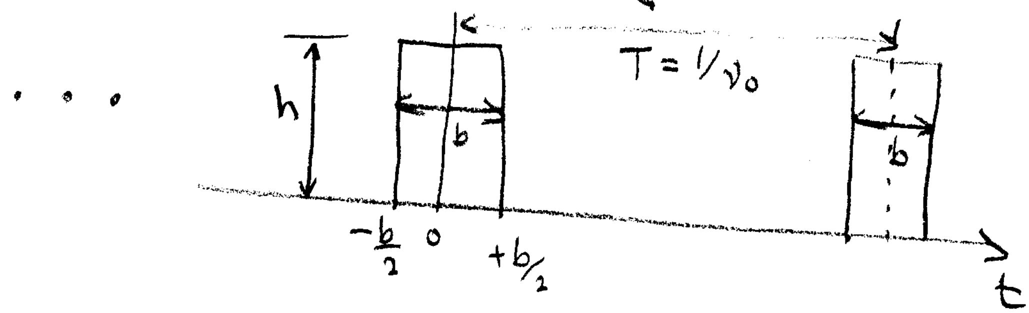
3.1 (a) Use the complex form:

$$f(t) = \sum_{-\infty}^{\infty} D_n e^{2\pi i n \nu_0 t}, \quad \text{where } D_n = \frac{1}{T}\int f(t)\, e^{-2\pi i n t/T}\, dt$$

to calculate D_n for a square wave as shown.



(b) Show that this is the same as that calculated from

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(2\pi n \nu_0 t) + B_n \sin(2\pi n \nu_0 t)$$

(c) Plot $\frac{D_n}{D_0}$ to show how the amplitudes of the higher order terms diminishes for 2 cases, $b = T/8$ and $b = T/4$.

(d) From (c) you should note that the bulk of the significant terms reside in the main lobe (between $n = 0$ + $n = n_0$ where $D_n \to 0$ for the first time). Based on this observation derive an expression for the number of terms (n_0) needed to capture the bulk of the signal.

Sol'n

(a) $D_n = \frac{1}{T}\int_T f(t)\, e^{-2\pi i n \nu_0 t}\, dt = \frac{h}{T}\int_{-b/2}^{b/2} e^{-2\pi i n \nu_0 t}\, dt$
(for $n \neq 0$)

$$= \frac{h\nu_0}{-2\pi i n \nu_0}\left(e^{-2\pi i n \nu_0 b/2} - e^{2\pi i n \nu_0 b/2}\right)$$

$$= h b \nu_0 \frac{\sin(\pi n b \nu_0)}{\pi n b \nu_0} = h b \nu_0 \operatorname{sinc}(\pi n b \nu_0)$$

$= D_{-n}$ since $\operatorname{sinc}(x)$ is an even function.

$$D_0 = \frac{1}{T}\int_T f(t)\, dt = \frac{hb}{T} = h b \nu_0 .$$

$$\therefore f(t) = \sum_{n=-\infty}^{\infty} h b \nu_0 \operatorname{sinc}(\pi n b \nu_0)\, e^{2\pi i n \nu_0 t}$$

(b) Expanding out we have

from $\sum_{n=1}^{\infty}$

$$f(t) = h b \nu_0 + h b \nu_0 \left[\operatorname{sinc}(\pi b \nu_0)\left(\cos 2\pi \nu_0 t + i \sin 2\pi \nu_0 t\right) + \ldots \right]$$

$$+ h b \nu_0 \left[\operatorname{sinc}(\pi b \nu_0)\left(\cos 2\pi \nu_0 t - i \sin 2\pi \nu_0 t\right) + \ldots \right]$$

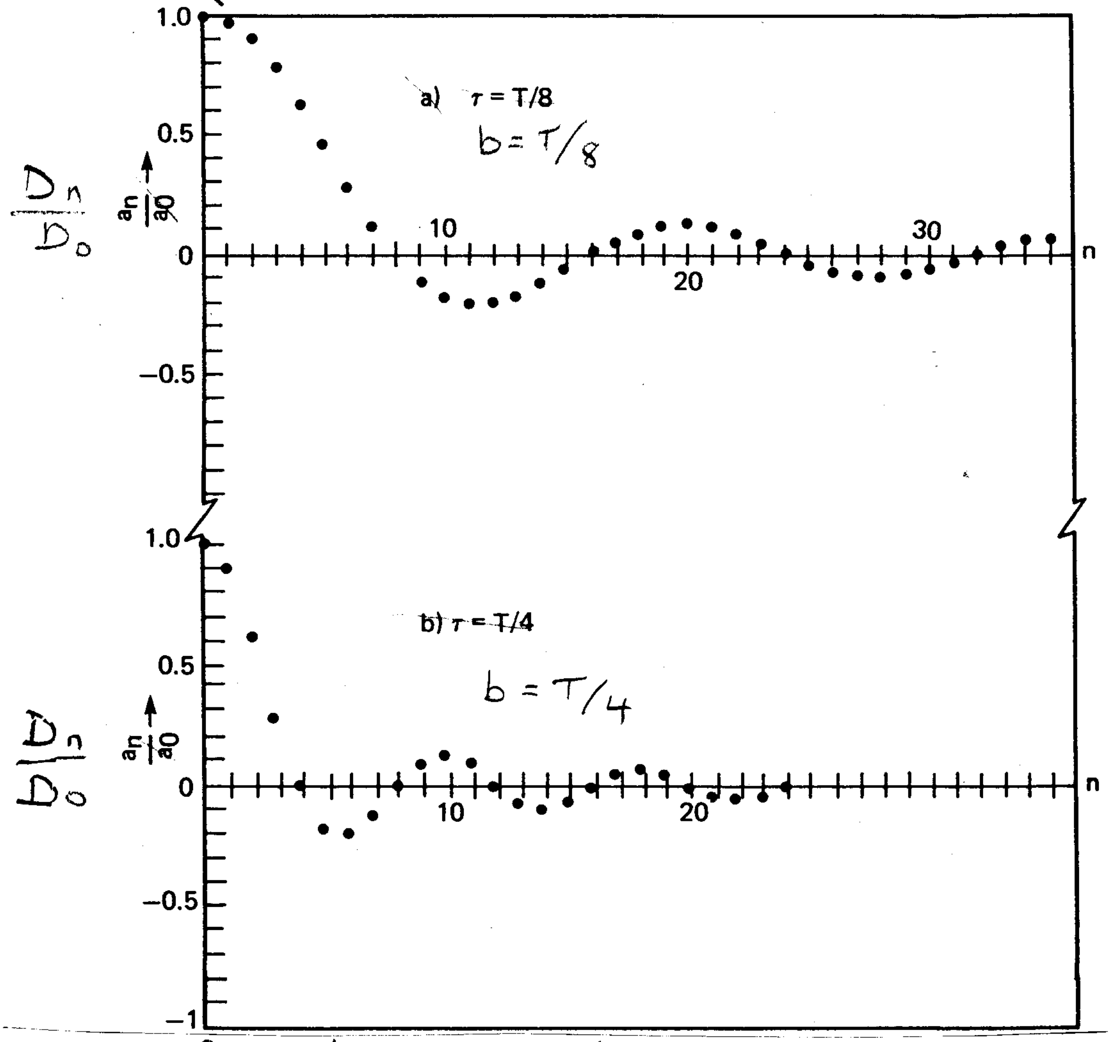
from $\sum_{n=-1}^{-\infty}$

$$= h b \nu_0 + 2 h b \nu_0 \sum_{n=1}^{\infty} \operatorname{sinc}(\pi n b \nu_0) \cos(2\pi n \nu_0 t)$$

which is what we found before when we calculated $A_n + B_n$.

QED

(c) $\dfrac{D_n}{D_0} = \dfrac{\cancel{h b \nu_0}}{\cancel{h b \nu_0}}\,\mathrm{sinc}\,(\pi n b \nu_0)$



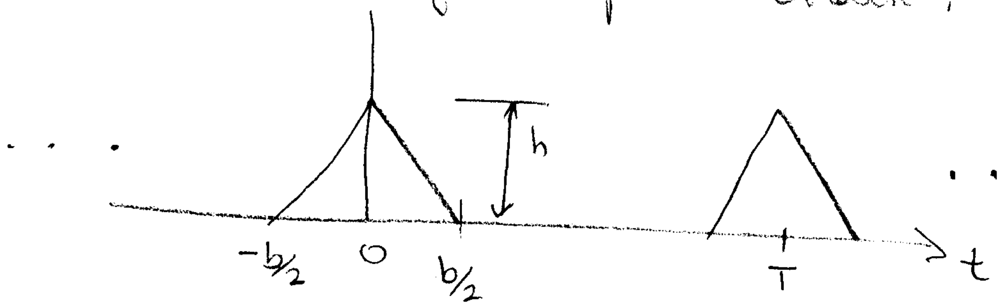
(d) The sinc function goes to 0 when

$\pi n b \nu_0 = \pi$, ie $n = \dfrac{1}{b\nu_0} = T/b \equiv n_0$

Thus, for the same period, T, a train of short pulses requires more terms than a train of long pulses.

This is a typical occurance worth noting.

3.2. (a) Use the complex form to calculate D_n for a triangular pulse train:



$T = 1/\nu_0$

(b) Plot D_n/D_0 for $b = T/8$ + $b = T/4$

(c) Derive an expression for the number of terms (n_0) needed to capture the bulk of the signal. Compare to the square pulse train.

Sol'n :

(a) $D_n = \frac{1}{T}\int_T f(t)\, e^{-2\pi i n \nu_0 t}\, dt$

$= \frac{h}{T}\int_{-b/2}^{0} \frac{2}{b}\left(t+\frac{b}{2}\right) e^{-2\pi i n \nu_0 t}\, dt$ $\leftarrow$ $= \frac{h}{T}\int_0^{b/2} \frac{2}{b}\left(\frac{b}{2}-t'\right) e^{2\pi i n \nu_0 t'}\, dt'$ where $t' = -t$

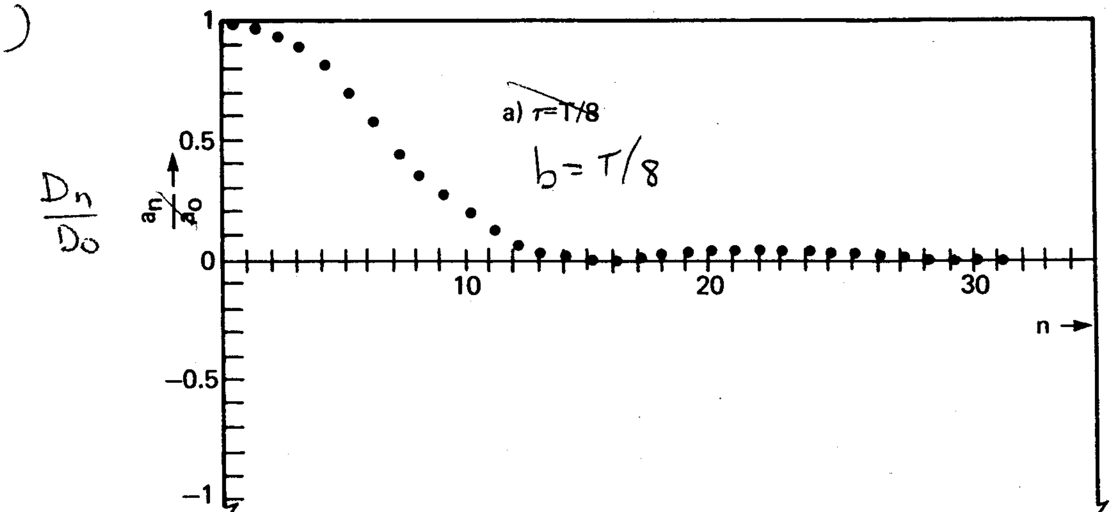
$+ \frac{h}{T}\int_0^{b/2} \frac{2}{b}\left(\frac{b}{2}-t\right) e^{-2\pi i n \nu_0 t}\, dt$

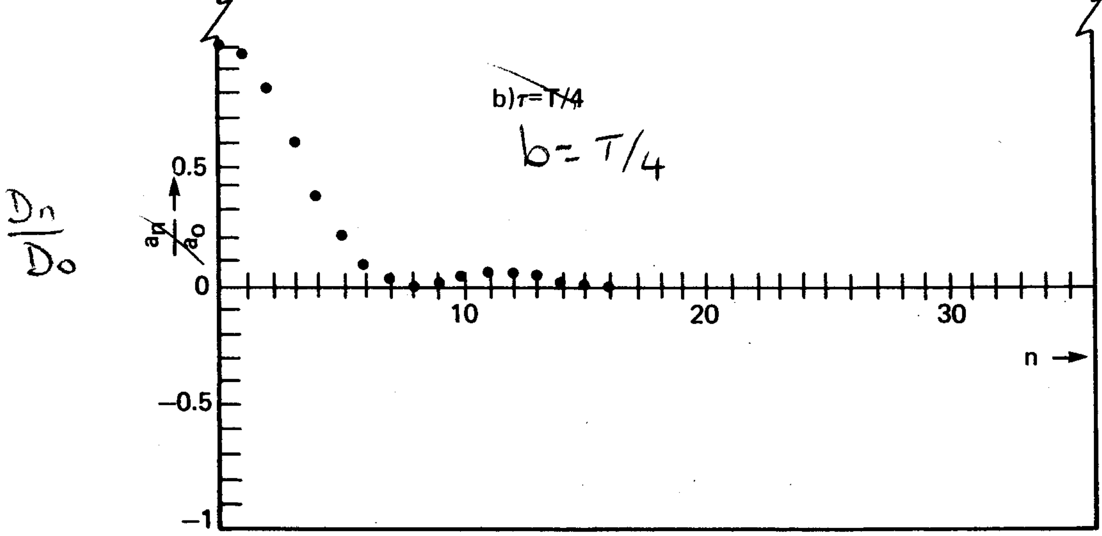
$= \frac{2h}{bT}\int_0^{b/2} \left(\frac{b}{2}-t\right)\left(e^{2\pi i n \nu_0 t} + e^{-2\pi i n \nu_0 t}\right) dt$

$= \frac{4h}{bT}\int_0^{b/2} \left(\frac{b}{2}-t\right) \cos(2\pi n \nu_0 t)\, dt$

$= \frac{hb}{2T}\, \text{sinc}^2\left(\frac{\pi n b}{2T}\right)$ where $T = 1/\nu_0$

(b)





(c)

The Sinc function → 0 when

$$\frac{\pi n b}{2T} = \pi, \text{ ie } n = \frac{2T}{b} \equiv n_0$$

Note that, compared to the square pulse train, the main lobe falls off more slowly but the side lobes are smaller.