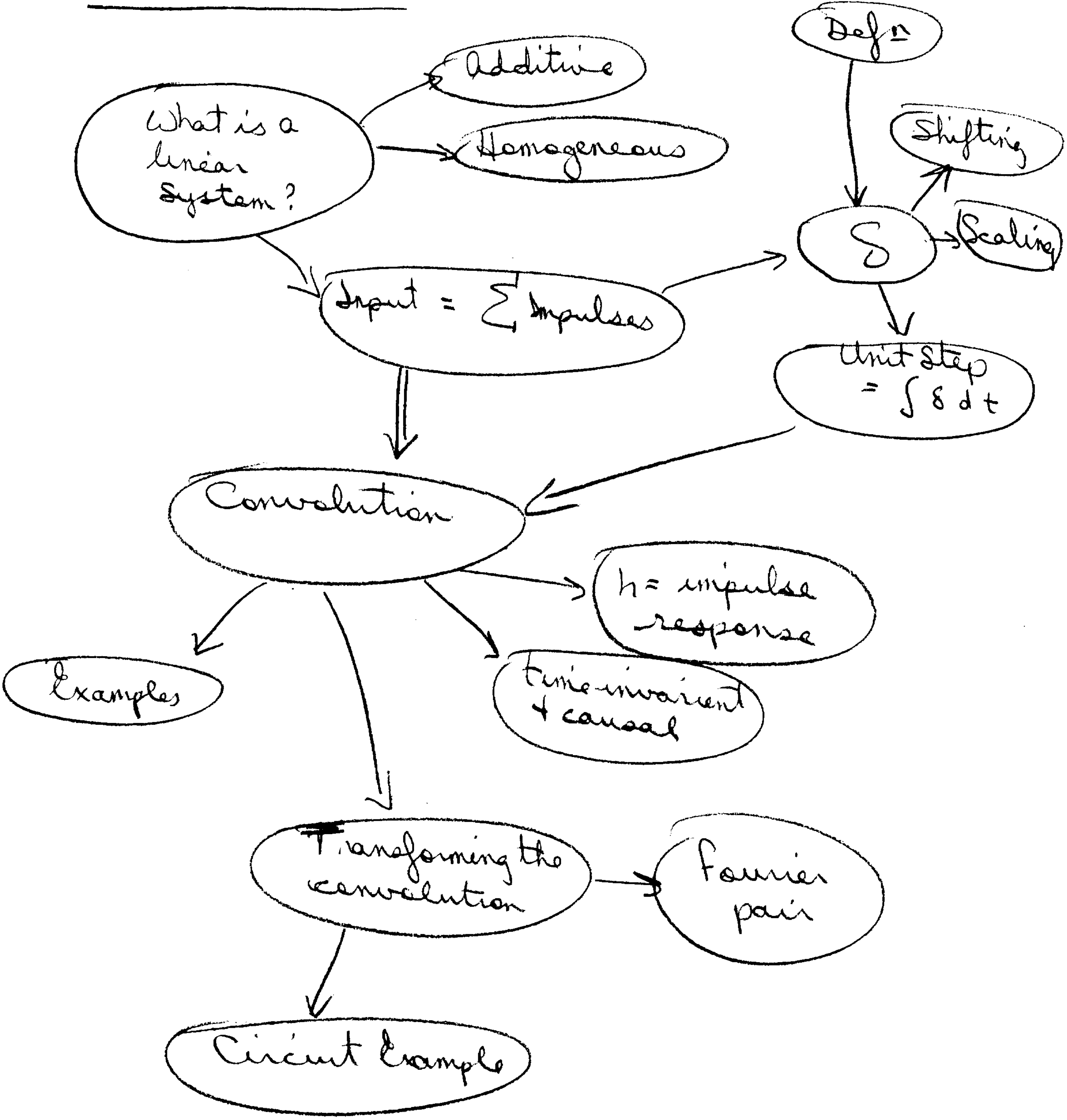
Chapter 2 - Linear Systems

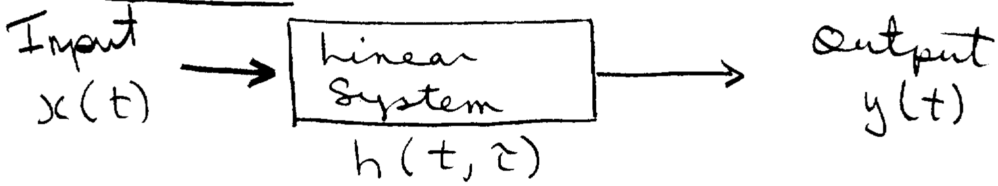
2.0 Overview



Chapter 2 - Linear Systems

(following Peebles 8.1)

2.1 Definition



The system "operates" on $x(t)$ to give $y(t)$

ie $y(t) = L[x(t)]$
$\uparrow$ general operator

The system is linear if response to sum of inputs

$$\left(\text{ie } y(t) = L\left[\sum_{n=1}^{N} \alpha_n x_n(t)\right]\right)$$

is the same as the sum of responses to the individual inputs

$$\left(\text{ie } y(t) = \sum_{n=1}^{N} \alpha_n L[x_n(t)] = \sum_{n=1}^{N} \alpha_n y_n(t)\right)$$

Linear = additive $(L\Sigma x = \Sigma L x)$
+ homogeneous $(L(\alpha x) = \alpha L(x))$

Any input can be regarded as a collection of impulses, ie

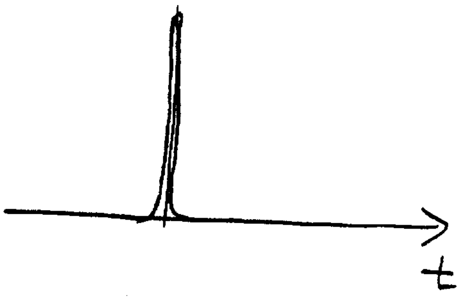
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

But before we explore this, we need to look at the δ-function.

2.2 The Delta Function, $\delta(t)$

$\delta(t)$ is the delta function

$$\delta(t) = \infty \quad \text{when } t = 0$$
$$= 0 \quad \text{when } t \neq 0$$

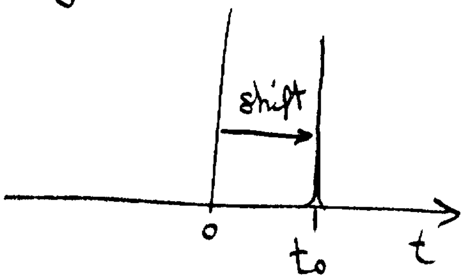


$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Shifting: Replace t by $t - t_0 \Rightarrow dt = d(t-t_0)$

$$\therefore \int_{-\infty}^{\infty} \delta(t)\,dt \Rightarrow \int_{-\infty}^{\infty} \underset{\uparrow \text{ or equally } \delta(t_0 - t)}{\delta(t-t_0)}\,dt = 1$$

where $\delta(t-t_0) = \infty$ when $t = t_0$
$= 0$ when $t \neq t_0$

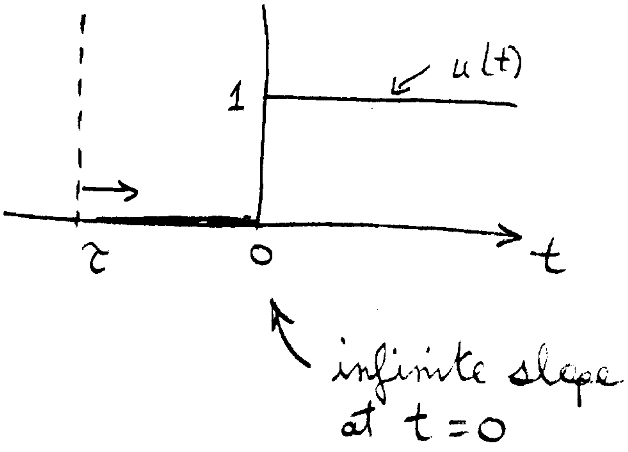


Unit Step

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) \leftarrow \text{unit step}$$

$$\Downarrow$$

$$\delta(t) = \frac{du(t)}{dt}$$



$u(t) = 1$ when $t > 0$
$= 0$ when $t < 0$
$=$ undefined when $t = 0$
but most choose $\frac{1}{2}$.

Note: The δ function is not really a function at all in the strict mathematical sense. We won't let that bother us.)

Also, from Fante, Chapter 2:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0) \qquad \text{where f is some well-behaved function.}$$

(Proof should be obvious)

Fante also shows:

$$\int_{-\infty}^{\infty} f(t)\,\delta(at-b)\,dt = \frac{1}{|a|}\, f(b/a) \quad \Longleftarrow \text{Scaling + Shifting (Useful result)}$$

<u>Proof</u>:

Let $x = at$ $\Rightarrow \displaystyle\int_{-\infty}^{\infty} f(x/a)\,\delta(x-b)\,\frac{dx}{a}$ if $a>0$

$$\Rightarrow \int_{\infty}^{-\infty} f(x/a)\,\delta(x-b)\,\frac{dx}{a} \quad \text{if } a<0$$

$$\text{ie} \Rightarrow \frac{1}{|a|}\int_{-\infty}^{\infty} f(x/a)\,\delta(x-b)\,dx$$

Now Let $y = x-b$

$$\Rightarrow \frac{1}{|a|}\int_{-\infty}^{\infty} f\left(\frac{y+b}{a}\right)\delta(y)\,dy$$

$$= \frac{1}{|a|}\, f(b/a)$$

<u>QED</u>

Watch those limits:

$$\begin{aligned} \int_{t_1}^{t_2} f(t)\,\delta(t-a)\,dt &= f(a), \quad t_1 < a < t_2 \\ &= \tfrac{1}{2} f(a), \quad t_1 = a \text{ or } t_2 = a \\ &= 0 \quad a < t_1 \\ &= 0 \quad a > t_2 \end{aligned}$$

2.3 The Convolution Integral

So we can replace $x(t)$ by this collection of impulses to get:

$$y(t) = \underset{\substack{\uparrow \\ \text{operates on} \\ \text{variable } t}}{L}[x(t)] = L\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} x(\tau)\,L[\delta(t-\tau)]\,d\tau$$

So if we know the system response to an impulse ($L[\delta(t)]$), then we can construct the system response to any input (as long as the system is linear).

This is a great simplifying feature.

Lets call this impulse response $h(t,\tau)$

Thus $y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t,\tau)\,d\tau$.

Now, for time invariant systems (ie you get the same response when you kick the system now vs. later)

$$h(t,\tau) = h(t-\tau)$$

$$\left(\text{ie}\quad \begin{aligned} \delta(t) &\Longrightarrow h(t) \\ \delta(t-\tau) &\Longrightarrow h(t-\tau) \end{aligned}\right)$$

Thus $L[\delta(t-\tau)] = h(t-\tau)$

This is another great simplifying feature – you only have to determine h once.

So we have:

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

This is called the convolution integral and is often written

$y(t) = x(t) * h(t)$ as a shorthand notation

[By the way, by a change of variables, you can write this as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$
]

So whether $x(\tau)$ (the input signal) is

- a pulse
- a pure sine wave
- a bunch of sine waves
- a square wave
- noise
- audio
- video
- a data stream
- whatever

and the system is

- an electronic circuit
- an instrument or probe
- a complex mechanical system
- a brain
- whatever

all we need is the impulse response to the system, h, and the data stream (input), x, and we can compute the integral to calculate the output.

2.4 Causal Systems

Physically we know that cause must come before effect, so

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau,$$

ie $\int_{t}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = 0$ which is automatically true since

$$h(t-\tau) = 0 \quad \text{when } \underset{\text{now}}{t} < \tau \leftarrow \text{integration variable.}$$

this is the "future" part of the integration.

(recall, $h(\tau) = 0$ when $\tau < 0$)

—

We will be dealing only with

$$\left\{\begin{matrix} \underline{\text{L}}\text{inear} \\ \underline{\text{T}}\text{ime-}\underline{\text{I}}\text{nvariant} \\ \underline{\text{C}}\text{ausal} \end{matrix}\right\} \text{ Systems}$$

or LTIC Systems

2.5 Transforming the Convolution Integral

Wouldn't it be nice if there was a way to simplify the calculation of $x * h$? There is!

Let's transform $y(t)$ as follows:

$$\boxed{Y(\nu) \equiv \int_{-\infty}^{\infty} y(t) e^{-2\pi j \nu t} \, dt}$$

$j \equiv \sqrt{-1}$
also
$i \equiv \sqrt{-1}$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right] e^{-2\pi j \nu t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\underbrace{\int_{-\infty}^{\infty} h(t-\tau) e^{-2\pi j \nu (t-\tau)} dt}_{H(\nu)} \right] e^{-2\pi \nu \tau} d\tau$$

$dt = d(t-\tau)$
Then let $z = t-\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-2\pi \nu \tau} d\tau \quad H(\nu)$$

$$= X(\nu) \, H(\nu)$$

Thus $Y(\nu) = X(\nu) \, H(\nu)$

$\uparrow$ transfer function of the system.

So in time space, we have a convolution integral

But in ν space, we have a multiplication.

$\uparrow$ we'll identify this later as frequency.

What is more, it turns out that we can transform $Y(\nu)$ back to $y(t)$ via

I'll prove this soon.

$$\boxed{y(t) = \int_{-\infty}^{\infty} Y(\nu) e^{2\pi j \nu t} d\nu}$$

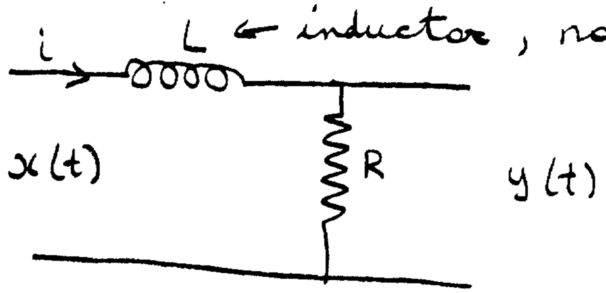
(note: there is no minus sign.)

$\uparrow$ this plus $Y(\nu) = \int y(t) \ldots$ Form Fourier Transform Pairs.

Fourier Transforms are so useful and ubiquitous that we need to study them in some detail before we can talk about signal processing and the like.

2.6 A Simple Example

L ← inductor, not the linear operator, L



$$\underset{\substack{\uparrow \\ \text{voltage in}}}{x(t)} = L\frac{di}{dt} + \underset{\substack{\uparrow \\ iR}}{y(t)} \leftarrow \text{voltage out}$$

Since $y = iR$,

$$\frac{dy}{dt} = R\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{1}{R}\frac{dy}{dt}$$

$$\therefore \quad x(t) = \frac{L}{R}\frac{dy}{dt} + y(t)$$

$$\therefore \int_{-\infty}^{\infty} x(t)\,e^{-2\pi j\nu t}\,dt = \frac{L}{R}\int_{-\infty}^{\infty}\frac{dy}{dt}e^{-2\pi j\nu t}\,dt + \int_{-\infty}^{\infty} y(t)\,e^{-2\pi j\nu t}\,dt$$

$$\text{ie} \quad X(\nu) = \frac{L}{R}2\pi j\nu\,Y(\nu) + Y(\nu)$$

↖ I'll prove this a bit later

$$\therefore \quad Y(\nu) = \frac{X(\nu)}{1+2\pi j\nu L/R} \equiv X(\nu)H(\nu)$$

$$\left(\text{ie } H(\nu) = \frac{1}{1+2\pi j\nu L/R}\right)$$

Thus, if we had $x(t)$ we could calculate $Y(\nu)$ and invert it to find $y(t)$.

For simple systems like this with simple inputs, you can directly integrate the PDE to get $y(t)$. But for more complex systems or complex inputs, the Transform Method is superior.

2.7 Recap

$$y(t) = L\Big[\overset{x(t)}{\overset{\shortparallel}{\sum_{n=1}^{N} \alpha_n x_n(t)}} \Big] = \sum_{n=1}^{N} \alpha_n L[x_n(t)]$$

$$= \sum_{n=1}^{N} \alpha_n y_n(t)$$

&

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$\Downarrow$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, \underset{\overset{|||}{h(t,\tau)}}{L[\delta(t-\tau)]}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{for LTIC sys.}$$

$$\left.\begin{aligned} &\text{Define: } Y(\nu) = \int_{-\infty}^{\infty} y(t)\, e^{-2\pi j \nu t} dt \\ &\text{will show: } y(t) = \int_{-\infty}^{\infty} Y(\nu)\, e^{2\pi j \nu t} d\nu \end{aligned}\right\} \text{F.T. pairs}$$

Get : $Y(\nu) = X(\nu) H(\nu)$

time space $\rightleftharpoons$ frequency space

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \rightleftharpoons Y(\nu) = X(\nu)\, H(\nu)$$

In a nutshell

Fourier Transforms help us solve the convolution integral which arises naturally in linear systems.