Chapter 3 Statistics for Small Systems

We study the microscopic world by looking at the "average" behaviour of the particles in ~~that~~ system. Hence we need to look at it from a statistical point of view.

Define P_s as the probability ~~that~~ a system is in state s.

$$\bar{f} \equiv \text{average of some function of the system}$$
$$= \sum_s P_s f_s$$

Example: coin flip: $\bar{f} = P_{heads} f_{heads} + P_{tails} + f_{tails}$

Choose $f_{head} = 1$, $f_{tail} = 0$

$$\therefore \bar{f} = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$$

Example: 6 sided die: $P_1 = P_2 \dots = P_6 = \frac{1}{6}$

Choose $f_n = n$ (the number that shows up on the roll).

$$\therefore \bar{f} = \sum_{n=1}^{6} P_n \cdot n = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \dots \frac{1}{6} \cdot 6$$
$$= 3.5$$

Note:
The answer isn't 3.0. Why?

Because "0" is not a possibility!

Some relations we can use to simplify expressions:

$$\overline{f+g} = \bar{f} + \bar{g} \quad , \quad \overline{cf} = c\,\bar{f}$$

Proof:

$$\overline{f+g} = \sum_n P_n \cdot (f_n + g_n) = \sum_n P_n f_n + \sum_n P_n g_n = \bar{f} + \bar{g}$$

$$\overline{cf} = \sum_n P_n \cdot c f_n = c \sum_n P_n f_n = c\,\bar{f}$$

Worked problems:

Do 3.1 of Stowe
3.3
3.4

Do 1.1
1.2
1.3
1.5
1.6
1.8

Binomial Distribution - Probabilities for Sys. of more than one element

Let p = prob. of success
q = " " failure

$$p+q=1 \quad \Rightarrow \quad q=1-p$$

If have 2 identical elements: ← necessary?

$$(p_1+q_1)(p_2+q_2)=1$$

all the possibilities

$$= \underbrace{p_1p_2 + p_1q_2 + q_2p_1 + q_1q_2}$$

p_1 AND p_2 — p_1 AND (NOT q_2)

If $p_1=p_2$, $q_1=q_2$

$$\Rightarrow (p+q)^2 = p^2 + 2pq + q^2$$

← 1 way of getting 2 tails

1 way of getting 2 heads

2 ways of getting 1 head + 1 tail

3 elements:

$$(p+q)^3 = p^3+3p^2q+3pq^2+q^3$$

etc.

In general, for N identical elements

$$(p+q)^N = \sum_{n=0}^{N} \underbrace{\frac{N!}{n!(N-n)!}} \underbrace{p^n q^{N-n}} = 1$$

← probability that n elements satisfy a criterion + the rest do not.

# of different configurations of N elements for which n ~~satisfy~~ a criterion. (binomial coefficient)

Prob. of being in a given state:

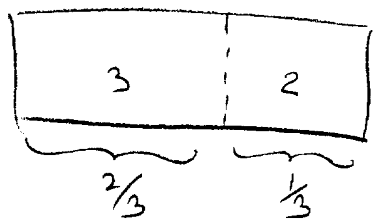
$$\boxed{P_N(n) = \frac{N!}{n!(N-m)!}\, p^n q^{N-n}}$$

say:
P N choose n

Example

5 air molecules in an empty room.

What is prob. of 2 in front 1/3, 3 in back.



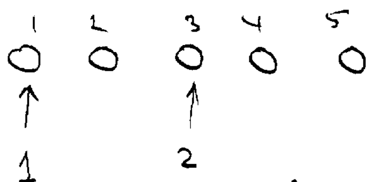
$$P_5(2) = \frac{5!}{2!\,3!}\, p^2 q^3$$

$$= \frac{5!}{2!\,3!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$$

$$= \frac{5 \times 4 \times \cancel{3!}}{2!\,\cancel{3!}} \cdot \frac{1}{9} \cdot \frac{8}{27}$$

$$= 10 \times \frac{8}{9 \times 27} = \frac{80}{243}$$

10 ways ~~that~~ 2 molecules out of 5 could be in front.



$1 \cdot 4$ = # of ways molecule 1 + one of the other 4 could be in front.

$\Downarrow$

$5 \cdot 4$ ways in all.

But can't distinguish molecules so configuration double up (ie 1+3 is same as 3+1)

$\therefore \frac{5 \cdot 4}{2} = 10$ unique ways.

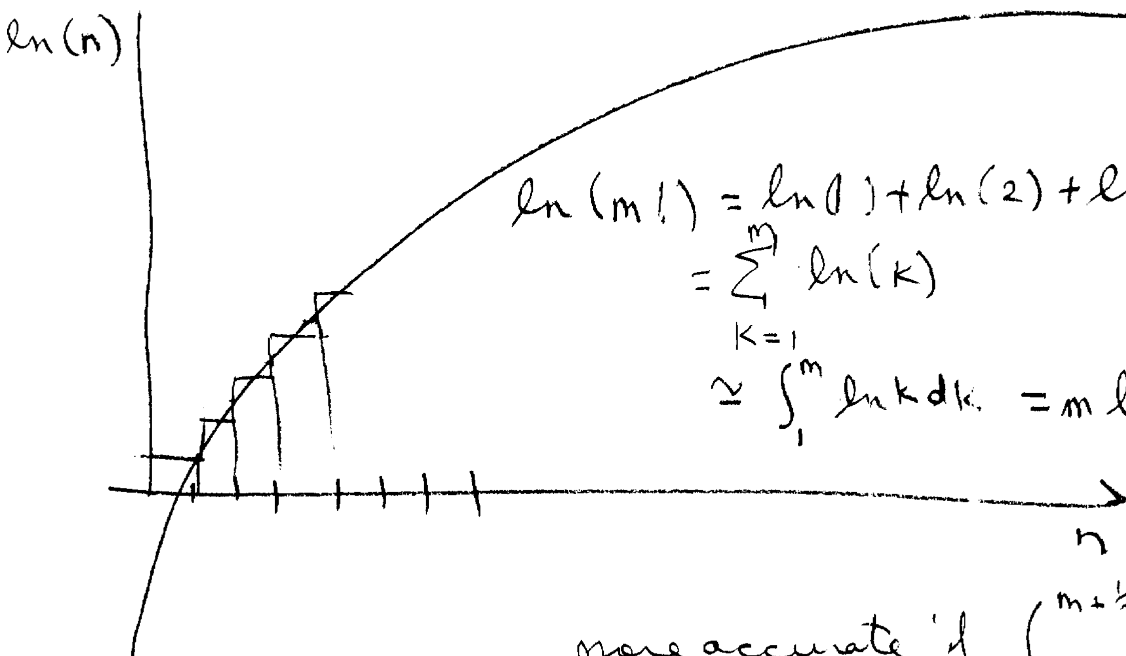
Stirling's formula

$N!$ is difficult to calculate for large N.

$\therefore$ Use Stirling's approximation:

$$\ln(m!) \simeq \underbrace{m \ln m - m}_{\text{the bulk of the value.}} + \underbrace{\tfrac{1}{2}\ln(2\pi m)}_{\text{small correction}}$$

for $m = 10$, error $= 1\%$.
Gets smaller as $m \uparrow$.



$$\ln(m!) = \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(m)$$
$$= \sum_{k=1}^{m} \ln(k)$$
$$\simeq \int_1^m \ln k \, dk = m\ln(m) - m$$

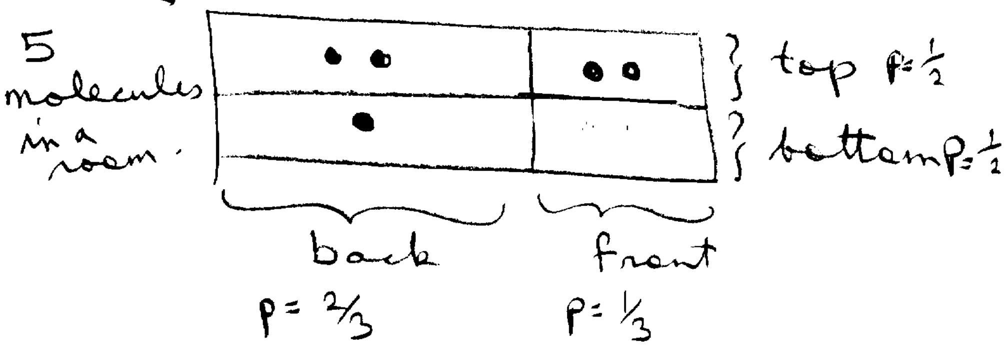
more accurate if $\int_{\frac{1}{2}}^{m+\frac{1}{2}}$

Statistically Independent Behaviours

If we have 2 independent criteria:

$$P_{ij} = P_i P_j$$

Example:



What is Prob of 2 molecules in front + 4 in top half?

$$P_5(2,4) = P_5(2)\, P_5'(4)$$

$$= \frac{5!}{2!\,3!}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^3 \cdot \frac{5!}{4!\,1!}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^1$$

$$= \frac{80}{243} \cdot \frac{5}{32} = 0.051$$

In general, if events are independent, can simplify expressions:

Eg: $$\bar{f} = \sum_{i,j} P_{ij} f(i) = \sum_{i,j} P_i P_j f(i) = \underbrace{\sum_j P_j}_{=1} \sum_i P_i f(i)$$

$$= \sum P_i f(i)$$

$\sim$ $\overline{fg} = \bar{f}\,\bar{g}$ if f + g are independent.

Worked Problems:

Do 3.5
3.7
3.8
3.9
3.10
3.11
3.12
3.13
3.15